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Invariant measure and universality of the 2D Yang-Mills Langevin dynamic (II)

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Based on joint work with Hao Shen: `arXiv:2302.12160`

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Stochastic Analysis meets QFT - critical theory
Münster

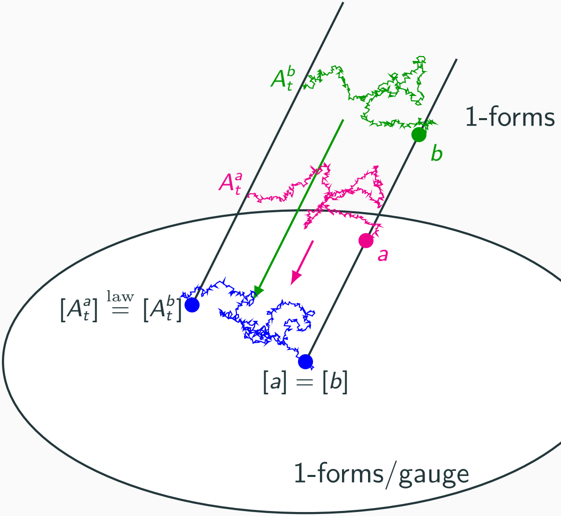
Overview

1. Identification of limit
2. Invariant measure
3. Corollaries and conclusion

Identification of limit

$$\partial_t A^a = -\tfrac{1}{2}\nabla S_{\mathrm{YM}}(A^a) + \mathrm{d}_{A^a}\,\mathrm{d}A^a + CA^a + \xi, \qquad A_0^a = a.$$

From [CCHS]:



Dynamic universality

From Hao's talk:

Theorem

On $\mathbb{T}^2$, the discrete Langevin dynamic for any 'nice' lattice YM model converges to

$$\partial_t A = \Delta A + A\partial A + A^3 + \bar{C} A + \xi$$

for some $\bar{C} \in L(\mathfrak{g}, \mathfrak{g})$.

Question: $\bar{C} = C$ for gauge covariant constant C?

- Computable in principle, but very lengthy.
- We are **not** allowed to renormalise discrete dynamic.

We show C is **unique** gauge covariant constant.

- Strengthens 2D continuum result.

Uniqueness of C

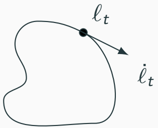
Consider $\bar{C} \neq C \in L(\mathfrak{g}, \mathfrak{g})$ and

$$\partial_t A = \Delta A + A^3 + A\partial A + \xi + \bar{C}A \,, \quad A_0 = 0 \,,$$
$$\partial_t B = \Delta B + B^3 + B\partial B + \xi + \bar{C}B \,, \quad B_0 = 0^g \,.$$

Theorem

There exists a loop $\ell \in \mathcal{C}^\infty(S^1, \mathbb{T}^2)$ such that for all $t > 0$ sufficiently small, there exists $g \in \mathcal{C}^\infty(\mathbb{T}^2, G)$ for which

$$|\mathbb{E}W_\ell(A(t)) - \mathbb{E}W_\ell(B(t))| \gtrsim t^2 \,.$$



Wilson loop: $W_\ell(A) = \mathrm{Tr}\,\mathrm{hol}(A, \ell) \in \mathbf{C}$, where $\mathrm{hol}(A, \ell) = y_1 \in G$

$$\mathrm{d}y_t = y_t \,\mathrm{d}\langle A(\ell_t), \dot{\ell}_t \rangle \,, \quad y_0 = I \in G \,.$$

Lemma: $a \sim b \Rightarrow W_\ell(a) = W_\ell(b)$.

Compare: if $\bar{C} = C$ then $|\mathbb{E}W_\ell(A(t)) - \mathbb{E}W_\ell(B(t))| \lesssim t^M$ for any $M > 0$.

Step 3: identification of limit

Two cases:

1. Abelian (topological).
2. Semi-simple (geometric)

Abelian:

- $G = \mathrm{U}(1)$, $\mathfrak{g} = i\mathbb{R}$, one can show $C = 0$.
- $a \sim b \Leftrightarrow \exists g \colon \mathbb{T}^2 \to \mathrm{U}(1),\ b = a - \mathrm{d}gg^{-1}$.
- For $\bar{C} \neq 0$,

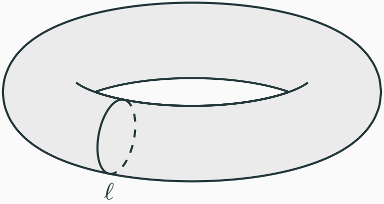
$$\partial_t A = \Delta A + \xi + \bar{C} A\ , \quad A(0) = 0\ ,$$
$$\partial_t B = \Delta B + \xi + \bar{C} B\ , \quad B(0) = 0^g = -\,\mathrm{d}gg^{-1}\ .$$

- $\Rightarrow B = -e^{t(\Delta + \bar{C})}\,\mathrm{d}gg^{-1} + A$.

Abelian case, $G = \mathrm{U}(1)$

- **Key:** while $A - \mathrm{d}gg^{-1} \sim A$, there exists g such that $A - \delta\,\mathrm{d}gg^{-1} \not\sim A$ for $\delta \ll 1$ (gauge orbit $[A]$ disconnected).

- **Non-contractible loop:** $\ell\colon [0,1] \to \mathbb{T}^2, \ell(x) = (x, 0)$.



- Take $g(x,y) = e^{i2\pi x} \Rightarrow -\mathrm{d}gg^{-1} = (-i2\pi, 0) \rightsquigarrow g$ lifts along ℓ to **non-contractile loop** in $\mathrm{U}(1)$.
- $B_1(t) = -i2\pi e^{t\bar{C}} + A_1(t) = i2\pi(1 + t\bar{C} + O(t^2)) + A_1(t) \Rightarrow$

$$|\mathbb{E}W_\ell(A(t)) - \mathbb{E}W_\ell(B(t))| = \left|\mathbb{E}e^{\int_\ell A(t)} - \mathbb{E}e^{\int_\ell B(t)}\right| \gtrsim t\ .$$

(Need torus, result not true on simply connected manifold, e.g. $\mathbb{R}^2$.)

Semi-simple case

Strategy: short time expansions.

By applying gauge transform, reduce problem to showing

$$|\mathbb{E}W_\ell(A(t)) - \mathbb{E}W_\ell(B(t))| \gtrsim t^2 ,$$

where

$$\partial_t A = \Delta A + A^3 + A\partial A + \xi + \bar{C}A , \qquad A(0) = 0$$
$$\partial_t B = \Delta B + B^3 + B\partial B + \xi + \bar{C}B + c\,\mathrm{d}gg^{-1} , \qquad B(0) = 0$$

for $\bar{C}, c \in L(\mathfrak{g}, \mathfrak{g})$, $c \neq 0$, and

$$\partial_t g = \text{parabolic PDE involving } B.$$
$$g(0) = \text{suitably chosen.}$$

Euler estimate for SPDE

Lemma

Let $h = \mathrm{d}g(0)g(0)^{-1}$. *Then*

$$B(t) = \Psi(t) + hO(t) + O(t^{2-\kappa}) ,$$

where Ψ *is explicit and* $hO(t)$ *is linear in* h *and independent of* $B(t)$. *Likewise*

$$A(t) = \Psi(t) + O(t^{2-\kappa}) .$$

In particular,

$$B(t) - A(t) = hO(t) + O(t^{2-\kappa}) .$$

Order 4 expansion in modelled distributions.

Cf. [Davie '08, Friz–Victoir '08].

Euler estimate for $W_\ell = \operatorname{Tr}\operatorname{hol}$ in semi-simple case

$$W_\ell(A(t)) = \operatorname{Tr}\sum_{k=0}^{N}\int_0^1 \cdots \int_0^{t_{k-1}} \mathrm{d}\gamma_{t_k}\ldots\mathrm{d}\gamma_{t_1} + \text{error}$$

with $\gamma\colon [0,1] \to \mathfrak{g}$ line integral of A_t (in sense of Young).

Lemma

$$\mathbb{E}W_\ell(A(t)) - \mathbb{E}W_\ell(B(t)) = L_t(h) + t^2\operatorname{Tr}((c\textstyle\int_\ell h)^2)/2 + O(t^{2+})$$

where $L_t(h)$ is linear in h.

NB. No order t term since $\operatorname{Tr}(\mathfrak{g}) = 0$ (cf. Abelian case).

- Chow–Rashevskii theorem for $G \times G \times \mathfrak{g} \Rightarrow \exists h, g, \tilde{g}$ such that $h = \mathrm{d}gg^{-1}, 4h = \mathrm{d}\tilde{g}\tilde{g}^{-1}$, $cX \neq 0$.
- L_t linear $\Rightarrow$ either g or $\tilde{g}$ gives $|\mathbb{E}W_\ell(A(t)) - \mathbb{E}W_\ell(B(t))| \gtrsim t^2$.

Remark: works for non-simply connected manifold.

Proves C is unique and identifies limit (universality).

Invariant measure

Invariant measure

Theorem (C.–Shen '23)

For $d = 2$, $[A]$ has a unique invariant probability measure μ on $\Omega/\sim$. Moreover, $\mu = \mu_{\mathrm{YM}_2}$, the YM measure on $\mathbb{T}^2$.

Steps in proof:

1. Find discrete approximation $\mu_{\varepsilon,\mathrm{YM}_2}$ of μ_{YM_2} such that discrete Langevin dynamic converges to SYM:

   $$\partial_t A = \Delta A + A\partial A + A^3 + CA + \xi \ .$$

   (Just finished.)

2. Moment bounds on discrete approximation $\mu_{\varepsilon,\mathrm{YM}_2}$.

Steps 1-2 combine in Bourgain's invariant measure argument.

Bourgain's argument

Suffices to show $A_t^\varepsilon := \varepsilon \log U_t$ does not blow up for $t \in [0,1]$ as $\varepsilon \downarrow 0$.

General strategy: cf. [Bourgain '94, Hairer–Matetski '16]

- Invariance of discrete dynamic:

$$\mathbb{P}\Big[\sup_{t\in[0,1]} \|A_t^\varepsilon\| > L\Big] \le K\mathbb{P}\Big[\sup_{t\in[0,1/K]} \|A_t^\varepsilon\| > L\Big]$$

- Take $K \gg L^{-q}$ for $q \gg 1$ fixed. (S)PDE estimate:

$$\|A_0^\varepsilon\| < L \Rightarrow \sup_{t\in[0,1/K]} \|A_t^\varepsilon\| \lesssim L\ .$$

- **Moment bounds:** **If** $\sup_{\varepsilon>0} \mathbb{E}[\|A_0^\varepsilon\|^p] > L] < \infty$ for all $p > 0$, **then**

$$\mathbb{P}\Big[\sup_{t\in[0,1]} \|A_t^\varepsilon\| > L\Big] \lesssim L^{q-p}\ .$$

- Take $p > q$ to conclude.

Difficulty: moment estimates do not hold for $\mu_{\varepsilon,\mathrm{YM}_2}$ due to gauge invariance.

Moment bounds

For $U\colon E_\varepsilon \to G$, find gauge-invariant measure of **non-flatness** $[\![U]\!]$ such that:

(a) $\|U^g\|_{\mathcal{C}^{0-}} \leq [\![U]\!]$ for discrete gauge transform g.

(b) $\mathbb{E}[\![U]\!]^p = O(1)$ uniformly in $\varepsilon > 0$ for $p \geq 1$.

Uhlenbeck compactness: for continuum A, $\exists g$ such that $\mathrm{d}^* A^g := \mathrm{div}(A^g) = 0$ (Coulomb/Landau gauge)

$$\mathrm{d}A = F_A - [A \wedge A]\,, \quad \mathrm{d}^* A = 0\,.$$

Elliptic regularity $\Rightarrow \|A\|_{W^{1,p}} \lesssim \|F_A\|_{L^p}$.

Can't apply directly: in regime where only $\|A\|_{\mathcal{C}^{0-}}$ is bounded.

(Cf. can't bound Brownian motion in $W^{1,p}$...)

Need Hölder-type norm

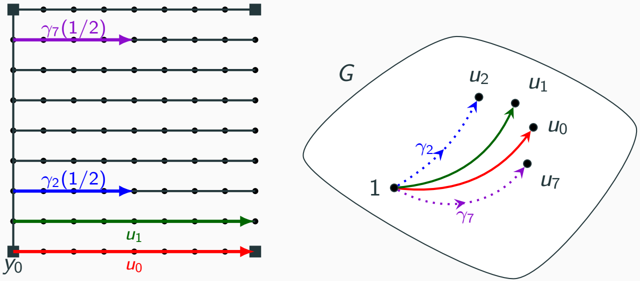
$$[\![U]\!] = \sup_r \frac{|\log U(r)| + \text{technical norm}}{|r|^{\frac{1}{2}-}}$$

(a) Gauge fixing - axial gauge

Idea: use mesoscopic (axial) and microscopic (Landau) gauges. From [C. '19]

Reason: PDE $\mathrm{d}A = F_A - [A \wedge A]$ is non-linear, need smallness to use ellipticity.

Axial gauge. Let $U\colon \Lambda_\varepsilon \to G$ be gauge field. Fix maximal tree T.



- $u_0, u_1, \ldots \in G$ given by U.
- Find γ_i connecting $1 \rightsquigarrow u_i$ in Lipschitz way (quantitative homotopy):

$$\varepsilon^{-1} |\log U_b^g| \lesssim \varepsilon^{-\frac{1}{2}-} [\![U]\!] + O(\varepsilon) \quad (\mathcal{C}^{-\frac{1}{2}-} \text{ control} \rightsquigarrow \text{suboptimal})$$

(a) Gauge fixing - Landau gauge

Zoom in **scale by scale:** if g defined on $\Lambda_{2\varepsilon}$, extend g to Λ_ε such that

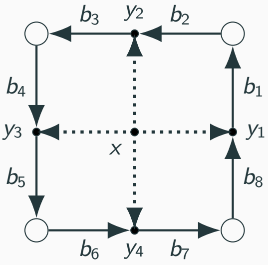
$$\log U^g_{b_2} = \log U^g_{b_3} = \frac{1}{2} \log U^g_{b_2 b_3} \quad \text{(likewise for } b_4, b_5, \ldots),$$

and $\sum_{i=1}^{4} \log U^g_{xy_i} \approx 0$

i.e. $\mathrm{d}^* \log U(x) \approx 0$.

(If $G = \mathrm{U}(1)$, can make $= 0 \pmod{2\pi}$.)

Approximately minimises $\sum_{i=1}^4 |\log U^g_{xy_i}|^2$.

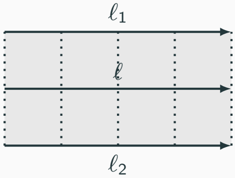


(a) Gauge fixing - Landau gauge

If ℓ is on lattice Λ_ε, then

$$\int_\ell \varepsilon^{-1} \log U^g = \underbrace{\frac{1}{2}\sum_{i=1,2}\int_{\ell_i} \varepsilon^{-1}\log U^g}_{\text{boundary term}} + \underbrace{\sum_{p\in\text{grey}} \log U(p)}_{\text{source term}: F_A} + \underbrace{\text{BCH errors}}_{[A\wedge A]}.$$

- boundary term: induction
- BCH errors: mild
- source term F_A: hardest part



Technical norm: q-var of anti-developments of $U \rightsquigarrow$ controls 'smearings' $\sum_p \log U(p)$ by gauge-inv. 'lasso smearings' $\sum_p \mathrm{Ad}_{U(\ell_p)} \log U(p)$ (Young sum/integral).

Outcome: If $\max_{b\in E_\delta} |\log U_b|$ is small, then $\exists g$: for all $b \in E_{\varepsilon<\delta}$

$$\varepsilon^{-1}|\log U_b^g| \lesssim \varepsilon^{0-} e^{C\log^2([\![U]\!]+1)} \quad (\mathcal{C}^{0-} \text{ control} \rightsquigarrow \text{optimal})$$

(b) Probabilistic estimates

To control $[\![U]\!]$ probabilistically, use **random walk representation** of $\mu_{\varepsilon,\mathrm{YM}_2}$:

$$U(r) \overset{\text{law}}{=} X_{\varepsilon^{-2}|r|}$$

where $X_0, X_1, \ldots$ is conditioned random walk on G with Brownian-like increments: $\mathbb{E}|\log X_i^{-1}X_{i+1}|^p \lesssim \mathbb{E}|B_{\varepsilon^2}|^p \sim \varepsilon^{p/2}$.

Ingredients:

- Uniform Gaussian tails $\mathbb{E}e^{\eta|\log X_{\varepsilon^{-2}t}|^2} < \infty$ for $\eta > 0$.
  - Rough path analysis of random walks.
- Uniform lower and upper bounds on density of $X_{\varepsilon^{-2}t}$ for $t > 0$ fixed.
  - Markov chain estimates (extension of [Hebisch–Saloff-Coste '93]).

Remark: only part requiring a priori knowledge of measure.

(But adaptable to some non-exactly solvable models [Chandra–C. '22].)

Corollaries and conclusion

Corollaries

Corollary (Long-time existence)

The Markov process $[A]$ survives for all time for all initial condition.

Proof: ergodicity theory of SPDEs. [Hairer–Mattingly '18, Hairer–Schönbauer '22]

Corollary (Gauge-fixed decomposition)

There exist a Gaussian free field Ψ and random function b such that $[\Psi + b] \sim \mu_{\mathrm{YM}}$ and $\mathbb{E}|b|^p_{\mathcal{C}^{1-\kappa}} < \infty$ for all $p \geq 1$, $\kappa > 0$.

Proof: decomposition of SPDE.

(Generalises main result of [C. '19].)

Corollary (Universality of measure)

Suppose $\mu_{\varepsilon,\mathrm{YM}_2}$ *is 'nice' approximation of* μ_{YM_2} *(e.g. Wilson, Villain, Manton actions).*

Then $\mu_{\varepsilon,\mathrm{YM}_2} \to \mu_{\mathrm{YM}_2}$ *in gauge-invariant f.d.d.*

Proof.

Moment bounds imply tightness of $\mu_{\varepsilon,\mathrm{YM}_2}$. Universality of dynamic + uniqueness of invariant measure identifies limit. □

Related work of [Driver '89]: Wilson action on $\mathbb{R}^2$.

Conclusion

- Link between YM Langevin dynamic and Euclidean QFT.
- Future work: other geometries for 2D YM?
  - Sphere $\mathbb{S}^2$: uniqueness of limit more subtle.
- Uniqueness of C in 3D?
  - Systematise Euler estimates..
  - Wilson loops need regularisation.
- Non-exactly solvable models? 2D YM–Higgs, 3D YM.
  - Progress on Abelian 2D YM–Higgs [Shen '21, Chandra–C. '22]

Thank you for your attention!