Non-commutative L^p spaces and Grassmann stochastic analysis

Francesco De Vecchi
Università degli Studi di Pavia

based on a joint work with Luca Fresta, Maria Gordina and Massimiliano Gubinelli

Stochastic Analysis meets QFT - critical theory,
June 12th 2023, Münster

Introduction

One of the first aims of Constructive Quantum Field Theory is the (more or less explicit) construction of models satisfying **Wightman axioms** or one of their equivalent formulations, for example **Osterwalder–Schrader axioms**.

Osterwalder–Schrader axioms describe the necessary and sufficient conditions such that a set of distributions $\mathcal{S}_n(x_1, \ldots, x_n)$ (where $x_i = (x_i^0, \ldots, x_i^{d-1}) \in \mathbb{R}^d$) are the Schwinger functions of some quantum field theory with a unique ground state.

This means, in non-formal sense, that there is a Hilbert space $\mathcal{H}$, a quantum field $\phi\colon \mathcal{D}\times\mathbb{R}^{d-1}\to\mathcal{L}(\mathcal{H})$ (where $\mathcal{D}\subset\mathbb{C}$ such that $x^0\in\mathcal{D}$ if $\Re(x^0)=0$, $\Im(x^0)\geqslant 0$ and $\mathcal{L}(\mathcal{H})$ is the set of linear operators on $\mathcal{H}$), and a ground state $\Omega\in\mathcal{H}$ such that

$$
\begin{aligned}
&\mathcal{S}_n(x_1,\ldots,x_n)=\\
&\quad\langle\phi(ix_1^0,x_1^1,\ldots,x_1^{d-1})\cdots\phi(ix_{n-1}^0,x_{n-1}^1,\ldots,x_{n-1}^{d-1})\phi(ix_n^0,x_n^1,\ldots,x_n^{d-1})\Omega,\Omega\rangle_{\mathcal{H}}
\end{aligned}
$$

where $0\leqslant x_1^0\leqslant x_2^0\leqslant\cdots\leqslant x_n^0$. In the bosonic case we have

$$
\phi_b(ix_1^0,x_1^1,\ldots,x_1^{d-1})\phi_b(ix_2^0,x_2^1,\ldots,x_2^{d-1})=\phi_b(ix_2^0,x_2^1,\ldots,x_2^{d-1})\phi_b(ix_1^0,x_1^1,\ldots,x_1^{d-1}).
$$

E. Nelson observed that many kind of Schwinger functions can be realized as the expectation of a (tempered-distribution) random field φ defined on $\mathbb{R}^d$:

$$
\mathcal{S}_n(x_1,\ldots,x_n)=\mathbb{E}_\nu[\varphi(x_1)\cdots\varphi(x_n)]
$$

Probabilistic quantization for bosons

Thanks to Nelson observation, the constructive quantum field theory for boson can be reduced to the problem of the definition of the measure ν of the random field φ.

In many cases of interest, the measure ν can be formally written as

$$\mathrm{d}\nu = '' e^{-S(\varphi)} \mathcal{D}\varphi '' = '' e^{-S_{\mathrm{int}}(\varphi)} '' \mathrm{d}\mu_{\mathrm{free}}$$

where $S = S_{\mathrm{free}} + S_{\mathrm{int}}$ is the (classical) action of the field φ, and μ_{free} is the Gaussian measure on the space of tempered distributions of the free field with variance S_{free}.

Thanks to this probabilistic formulation of the problem, in combination with renormalization techniques, it was possible to build many models of interacting bosons (see, e.g., [Simon,1973], [Glimm-Jaffe,1981], [Rivasseau,1991]).

Another probabilistic method for the construction of ν is **stochastic quantization**, first proposed by [Parisi-Wu, 1981] (see [Damgaard-Hüffel,1987]). It is based on the observation that $\nu \sim \Phi(t, \cdot)$ can be seen as the probability law of the invariant solution Φ to the SPDE

$$\frac{\partial \Phi}{\partial t}(t,x) = -\frac{\delta S}{\delta \varphi}(\Phi(t,x)) + \xi(t,x)$$

where $x \in \mathbb{R}^d$, t is an additional "computer time", $\frac{\delta S}{\delta \varphi}$ is the functional derivatives of the action S, and ξ is a $\mathbb{R}^{1+d}$ (space-times) white noise.

Thanks to the relatively recent advances in the analysis of SPDEs, the previous equation has been studied using different techniques: *regularity structures* [Hairer,2014], *paracontrolled distributions* [Gubinelli-Imkeller-Perkowski,2015], methods based on *renormalization group techniques* [Kupiainen,2016], and methods based on *rough path theory* [Otto-Weber,2019].

A final probabilistic method for building ν is based on a stochastic optimal control problem studied in [Barashkov-Gubinelli,2020].

Euclidean QFT for fermions

Osterwalder–Schrader axioms can also be used for quantum field theories including fermions. The most important difference of (Euclidean) fermionic fields ϕ_f is that they commute when evaluated at different points:

$$\phi_f(ix_1^0, x_1^1, \ldots, x_1^{d-1})\phi_f(ix_2^0, x_2^1, \ldots, x_2^{d-1}) = -\phi_f(ix_2^0, x_2^1, \ldots, x_2^{d-1})\phi_f(ix_1^0, x_1^1, \ldots, x_1^{d-1}).$$

This means that the Schwinger functions

$$\mathcal{S}_n(x_1, \ldots, x_n) = \langle \phi_f(ix_1^0, x_1^1, \ldots, x_1^{d-1}) \cdots \phi_f(ix_{n-1}^0, x_{n-1}^1, \ldots, x_{n-1}^{d-1})\phi_f(ix_n^0, x_n^1, \ldots, x_n^{d-1})\Omega, \Omega\rangle_{\mathcal{H}}$$

cannot be realized as expectations of (standard) random fields.

Our main aim is to find a probabilistic-like framework where to describe the fermionic QFTs.

Feynman-Kac formula, proposed in [Osterwalder-Schrader,1973/74], for fermions with action $S = S_{\rm free} + S_{\rm int}$ is

$$\begin{aligned} \mathcal{S}_n(x_1,\dots,x_n) &= \omega(\psi(x_1)\cdots\psi(x_n)) \\ &= \omega_{\rm free}(\,{}''e^{-S_{\rm int}(\psi_{\rm free})}\,{}''\psi_{\rm free}(x_1)\cdots\psi_{\rm free}(x_n)) \end{aligned} \qquad (1)$$

where $\psi_{\rm free}$ is the non-commutative random field having the "Gaussian" law related to the quadratic form $S_{\rm free}$. Other probabilistic-like constructions are discussed in [Fröhlich-Osterwalder,1974].

Formula (1), in combination with renormalization group methods, has been applied to many fermionic models in quantum field theory and condensed matter, see, e.g., [Gawedzki-Kupiainen, 1985], [Feldman-Magnen-Rivasseau-Sénéor,1986] [Lesniewski, 1987], [Benfatto-Gallavotti, 1990], [Disertori-Rivasseau, 2000], [Giuliani-Mastropietro-Porta, 2017], [Giuliani-Mastropietro-Rychkov, 2020], and see [Salmhofer, 2007], [Mastropietro, 2008] for some reviews.

Grassmann fields in C^* algebras

In [Albeverio-Borasi-D-Gubinelli,2022] we give C^*-algebra construction of fermions

We consider a C^*-algebra $\mathcal{M}$ which stays for the space of (bounded) random variables, i.e. $\mathcal{M}$ is a Banach algebra with a conjugate operation $\cdot^*$ such that for any $a \in \mathcal{M}$ we have

$$\|a\|_{\mathcal{M}}^2 = \|a a^*\|_{\mathcal{M}} = \|a^* a\|_{\mathcal{M}}.$$

The probability measure is here replaced by a positive state ω defined on $\mathcal{M}$, i.e. $\omega: \mathcal{A} \to \mathbb{C}$ is a linear map for which $\omega(a a^*) \geqslant 0$.

This kind of formulation of probability contains the standard (commutative) probability as a special case: given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we have that

$$\mathcal{A} = L^\infty(\Omega, \mathbb{C}) \quad \omega(\cdot) = \mathbb{E}_{\mathbb{P}}[\cdot].$$

In this contest a *Grassmann random field* X defined on the vector space V is a linear map

$$X\colon V\to\mathcal{M}$$

such that for any $v_1, v_2\in V$ we have

$$X(v_1)X(v_2)=-X(v_2)X(v_1).$$

We can extend $X\in\mathcal{L}(V,\mathcal{M})$ to a homomorphism $X\colon\Lambda V\to\mathcal{M}$, where ΛV is the exterior algebra generated by V equipped with the natural exterior product.

In this way, we interpret the elements $F\in\Lambda V$, i.e. $F=\sum_{k,\alpha_i} v_{\alpha_1}\wedge\cdots\wedge v_{\alpha_k}$, as the functions from the space of G. fields $\mathcal{M}$ (the space of random variables), i.e.

$$F(X):=X(F)=\sum_{k,\alpha_i} X(v_{\alpha_1})X(v_{\alpha_2})\cdots X(v_{\alpha_k}).$$

Suppose that V is equipped with a pre-Hilbert (real) scalar product $\langle \cdot, \cdot \rangle_V$ and let G be an antisymmetric operator on V with respect to $\langle \cdot, \cdot \rangle_V$.

In this setting, a Gaussian Grassmann random field X is a Grassmann random field such that, for any $v_1, \ldots, v_n \in V$ we have

$$\omega(X(v_1) \cdots X(v_{2n})) = \sum_{\mathcal{P} \in \{\text{perfect matching of } \{1, \ldots, 2n\}\}} (-1)^{\mathcal{P}} \prod_{(i,j) \in \mathcal{P}} \langle v_i, G v_j \rangle .$$

Topological requirement: It is useful to ask for the topological property

$$\|X(v)\|_{\mathcal{M}} \leqslant \sqrt{\langle v, v \rangle_V}$$

Lemma *For every antisymmetric and bounded $G: V \to V$ there exist a pair $(\mathcal{M}, \omega)$ and Grassmann Gaussian field X defined on V with correlation G.*

Idea of the proof: Suppose that $V = V_1 \oplus V_1$, and

$$G = C \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = CJ.$$

Consider the fermionic Fock space $\Gamma_f V$ and let $\mathcal{M} = \mathcal{L}(\Gamma_f V)$ with the state $\omega(a) = \langle a\Omega, \Omega\rangle$, where $\Omega \in \Gamma_f V$ is the ground state.

We can identify $\Gamma_f V$ with the (Hilbert closure) of ΛV, which is the exterior algebra generated by V equipped with the scalar product

$$\langle v_1 \wedge \ldots \wedge v_n, w_1 \wedge \ldots \wedge w_n\rangle = \det((\langle v_i, w_j\rangle_V)_{i,j=1,\ldots,n}).$$

Let λ, λ^* be the creation $\lambda\colon V \to \mathcal{L}(\Gamma_f V)$ and annihilation $\lambda^*\colon V \to \mathcal{L}(\Gamma_f V)$ operators on $\Gamma_f V$, i.e. for any $v, v_1, \ldots, v_n \in V$, $a = w_1 \wedge \cdots \wedge w_n \in \Gamma_f V$

$$\lambda(v)(a) = v \wedge v_1 \wedge \cdots \wedge v_n,$$

$$\lambda(v)^* a = \sum_{\ell=1}^{n} (-1)^{\ell-1} \langle v, w_\ell \rangle w_1 \wedge \cdots \wedge \not{w}_\ell \wedge \cdots \wedge w_n.$$

Then $X\colon V \to \mathcal{M} = \mathcal{L}(\Gamma_f V)$ can be defined as

$$X(v) := \lambda\left(C^{\frac{1}{2}} v\right) + \lambda^*\left(C^{\frac{1}{2}} J v\right).$$

A Grassmann stochastic process on V is a Grassmann random filed defined on $V_{\mathbb{R}}$ where

$$V_{\mathbb{R}} = L^2(\mathbb{R}_+) \otimes V.$$

A Gaussian Grassmann noise with covariance G (defined on V) is a Grassmann stochastic process Ξ on $V_{\mathbb{R}}$ equipped by the pre-Hilbert norm of the tensor product $L^2(\mathbb{R}_+) \otimes V$ and with covariance $\mathbb{I}_{L^2} \otimes G$.

We define also the concept of Grassmann Brownian motion as the "integral" of the noise Ξ with respect to the time

$$B_t(v) = \Xi(\mathbb{I}_{[0,t]} \otimes v).$$

This setting was enough to prove the existence and uniqueness of **additive noise, non-singular SDEs both in finite and infinite dimension**. Furthermore we were able to prove that the "invariant measure" of an additive noise SDE with gradient type drift $\partial U(\psi)$ is the associated Gibbs measure $e^{-2U(\psi)}$ (see [Albeverio-Borasi-D-Gubinelli,2022]).

In [D-Fresta-Gubinelli,2022] we apply a similar setting to singular (super-renormalizable) fermionic models. In this work we do not use a Langevin dynamic but a combination of Barashkov-Gubinelli stochastic variational method and the Polchinski flow.

An other work, using C^*-algebras for fermions, is [Chandra–Hairer–Peev,2023] where the authors develop a setting for "almost sure" analysis of non-commutative random variables via the notion of *locally C^*-algebras*, and they solve (locally in time) some singular SPDEs.

Consider the Grassmann field $(\Psi, \bar{\Psi})$ on $V = L^2(\mathbb{T}^2) \otimes \mathbb{R}^2$ with covariance

$$G = \begin{pmatrix} 0 & (-\Delta_{\mathbb{T}^2} + m^2)^{-1} \\ -(-\Delta_{\mathbb{T}^2} + m^2)^{-1} & 0 \end{pmatrix}.$$

In this case $\Psi, \bar{\Psi}$ cannot be identified with function from $\mathbb{T}^2 \rightarrow \mathcal{M} = \mathcal{L}(\Gamma_f V)$ but only with proper distribution. In analogy with the bosonic case we can define the Wick product $:\Psi(x)\bar{\Psi}(x):$ as an operator on $\Gamma_f V$, but $:\Psi(x)\bar{\Psi}(x): \notin \mathcal{M}$.

A similar problem arise in the definition of Ito integral. Let $H: \mathbb{R}_+ \times V \rightarrow \mathcal{M}$ a (predictable) bounded process then the Riemann sums

$$\sum_{t_i \in \pi} H_{t_i}(v)(B_{t_{i+1}}(v) - B_{t_i}(v))$$

do not usually converge in $\mathcal{M}$.

A classical extension of measure theory to the non-commutative setting was proposed by [Segal,1953]. In this case we consider a C^*-algebra (or better von Neumann algebra) $\mathcal{M}$ equipped with **a tracial state $\omega = \tau$**, namely a state for which

$$\omega(ab) = \tau(ab) = \tau(ba)$$

for any $a, b \in \mathcal{M}$. Using this kind of states we can define some new norms $\mathcal{M}$ as

$$\|a\|^p_{L^p_\tau(\mathcal{M})} = \tau(|a|^p). \qquad (2)$$

This permits to build a space $L^p_\tau(\mathcal{M})$ for any $p \geqslant 1$. In particular $L^\infty_\tau(\mathcal{M}) = \mathcal{M} \subset L^p(\omega)$, and for $a \in \mathcal{M}$ the norm reeds exactly as in formula (2).

In this kind of spaces is possible to obtain:

- Hölder inequality (and as a consequence $L = \bigcap_{1 \leqslant p < \infty} L^p(\omega)$ is an algebra);
- If $\{\mathcal{M}_t\}_{t \in \mathbb{R}_+}$ is an increasing filtration of C^*-subalgebras of $\mathcal{M}$ is possible to introduce a sort of conditional expectation $\tau_t \colon L^p_\tau(\mathcal{M}) \to L^p_\tau(\mathcal{M}_t)$;
- Hypercontractivity for Gaussian random variables (or fields) holds (see [Carlen-Lieb,1993]);
- Stochastic calculus for "Brownian motion" (or more generally continuous martingales) with Itô integral and the martingale representation theorem (see [Barnett-Streater-Wilde,1982]).

The main problem of this approach is that we are not able to build a Grassmann Brownian motion with respect to any tracial state τ, since the tracial nature of τ forces the covariance to be zero due to the contrasting requirements of symmetry and antisymmetry.

Tomita–Takesaki theory

Consider a von Neumann algebra $\mathcal{M}$ equipped with **a faithful state** ω, i.e. for any $a \in \mathcal{M}$

$$\omega(a^* a) \neq 0.$$

In this case is possible to define the **Tomita–Takesaki modular automorphism** $\sigma_t \colon \mathbb{R}_+ \times \mathcal{M} \to \mathcal{M}$, i.e. for $a \in \mathcal{M}$

$$\sigma_t(a) = \Delta^{-it} a \Delta^{it} \in \mathcal{M}$$

where Δ is a (unbounded self-adjoint) operator on the GNS representation of $(\mathcal{M}, \omega)$.

The Tomita–Takesaki automorphism "measures" the non-traciality of the state ω. Indeed let $\mathcal{M}_a \subset \mathcal{M}$ be the algebra of analytic elements of $\mathcal{M}$, i.e. $b \in \mathcal{M}_a$ if

$$t \in \mathbb{C} \mapsto \sigma_t(b) \in \mathcal{M}$$

is an entire map. Then for any $a, b \in \mathcal{M}_a$ we have

$$\omega(ab) = \omega(\sigma_{-i}(b)\, a).$$

In the case where ω is tracial $\sigma_t = \mathbb{I}_{\mathcal{M}}$ and $\Delta = \mathbb{I}_{\mathcal{H}}$.

Haagerup in [Haagerup,1979] proposed a standard way of building non-commutative $L^p_\omega(\mathcal{M})$ spaces in this non-tracial setting. Non-tracial L^p spaces have been extensively studied in the literature, in particular standard martingale inequalities are available, see, e.g., the work of [Pisier–Xu,1997], [Junge,2002], [Junge–Xu,2003], and hypercontractivity of some particular Gaussian factors has been proved by [Lee–Richard,2011].

The main problem of Haagerup L^p spaces is that they are not defined directly as a subset of operators in the GNS (or generic) representation of $(\mathcal{M}, \omega)$. This fact in particular implies that $L^p_\omega(\mathcal{M}) \cap L^q_\omega(\mathcal{M}) = \emptyset$ when $p \neq q$.

So we introduce the notion of twisted $\mathbb{L}^p_\omega(\mathcal{M})$ spaces. In the case where $p \in 2\mathbb{N}$ we can define the norm $x \in \mathcal{M}_a$

$$\|x\|^p_{\mathbb{L}^p} = \sup_{|\tau| \leqslant 1 - \frac{1}{2p}} \omega\Big(\sigma_{i\left(\tau - \frac{2p-1}{2p}\right)}(x)\sigma_{-i\left(\tau + \frac{2p-3}{2p}\right)}(x^*) \cdots \sigma_{i\left(\tau - \frac{3}{2p}\right)}(x)\sigma_{-i\left(\tau + \frac{1}{2p}\right)}(x^*)\Big).$$

This definition is a modification of the one proposed in [Majewski-Zegarlinski,1996] (who had considered only the norm $\tau = 0$).

There are some maps

$$T_\tau^{(p)} \colon \mathbb{L}^p_\omega(\mathcal{M}) \to L^p_\omega(\mathcal{M}), \quad |\tau| \leqslant (2p-1)/2p.$$

and we can write

$$\|x\|_{\mathbb{L}^p_\omega(\mathcal{M})} = \sup_{|\tau| \leqslant 1 - \frac{1}{2p}} \|T_\tau^{(p)}(x)\|_{L^p_\omega(\mathcal{M})}.$$

We have also

$$\mathbb{L}^p_\omega(\mathcal{M}) \subset \mathbb{L}^q_\omega(\mathcal{M}), \quad q \leqslant p$$

$$\mathcal{M}_a \subset \mathbb{L}^p_\omega(\mathcal{M}) \quad p \geqslant 1.$$

The Hölder inequality holds:

$$\|xy\|_{\mathbb{L}^r_\omega(\mathcal{M})} \leqslant \|x\|_{\mathbb{L}^p_\omega(\mathcal{M})} \|y\|_{\mathbb{L}^q_\omega(\mathcal{M})} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

The previous construction of Grassmann Gaussian random fields was not done with a faithful state ω.

Our proposal is to realize Grassmann Gaussian probability over $\mathbb{L}^p$ through a large class of non-Fock (and non-tracial) quasi-free states of the CAR algebra, and in particular by the Araki–Wyss factors [Araki-Wyss,1964].

Indeed, fix a (real) Hilbert space V, consider $\Gamma_a(V \oplus V)$, the operators

$$\gamma_\rho(f) = \lambda(\mu f \oplus 0) + \lambda^*(0 \oplus \mu^{-1} f), \quad f \in V, \; 0 < \mu < 1$$

and the von Neumann algebra

$$\mathcal{M} := \mathcal{M}(V, \mu) = \{\gamma_\mu(f), \quad f \in V\}''.$$

The Araki–Wyss factors have the properties

$$\{\gamma_\mu(f), \gamma_\mu(g)\} = \{\gamma_\mu^*(f), \gamma_\mu^*(g)\} = 0$$

$$\{\gamma_\mu^*(f), \gamma_\mu(g)\} = (\mu^2 + \mu^{-2})\langle f, g\rangle_V.$$

One can check that the Fock vacuum $\omega(\cdot) = \langle \Omega, \cdot\Omega\rangle_{\Gamma_a(V\oplus V)}$ is a faithful quasi-free state. The Tomita–Takesaki modular operator/automorphism is

$$\Delta = \Gamma_a(\rho^4\mathbb{I}_V \oplus \rho^{-4}\mathbb{I}_V), \quad \sigma_t(\gamma_\mu(f)) = \mu^{-4it}\gamma_\mu(f),$$

In particular the previous equality implies that $\gamma_\mu(f), \gamma_\mu^*(f) \in \mathcal{M}_a$.

Thanks to these factors we can defined Grassmann Gaussian field, by replacing λ, λ^* in the C^*-algebraic construction by γ_μ, γ_μ^* as follows

$$X(f) = \frac{1}{\mu^2 - \mu^{-2}} \left(\gamma_\mu \left(C^{\frac{1}{2}} J f \right) + \gamma_\mu^* \left(C^{\frac{1}{2}} f \right) \right)$$

which is well defined whenever $0 < \mu < 1$ (and thus $\Delta \neq \mathbb{I}_{\Gamma_a(V \oplus V)}$).

Since γ_μ, γ_μ^* are quasi-free, X is Gaussian, and we have

$$\sigma_{it}(X(f)) = \mu^{-4t} X(f)$$

and thus

$$\|X(f)\|_{\mathbb{L}^p} \lesssim \left(\sup_{|\tau| \leqslant 1 - \frac{1}{2p}} \mu^{-4\tau} \right) \|f\|_V.$$

Thanks to the previous construction we are able to:

- Prove Hypercontractivity for functionals of Gaussian random fields in $\mathbb{L}^p$ spaces;
- Generalize the notion of Brownian motion $B_t \in \mathcal{M}_a \subset \mathcal{M}$ (with an antisymmetric covariance $G = CJ$), satisfying $\|B_t - B_s\|_{\mathbb{L}^p} \lesssim_p |t-s|^{\frac{1}{2}}$;
- Define a conditional expectation $\omega_t\colon \mathbb{L}^p_\omega(\mathcal{M}) \to \mathbb{L}^p_\omega(\mathcal{M}_t)$, where $\mathcal{M}_t$ is the von Neumann algebra generated by B_s, B_s^* for $s \leqslant t$;
- Define the Itô integral $\int_0^t H_s \mathrm{d}B_s$, for $H\colon \mathbb{R}_+ \to \mathbb{L}^2_\omega(\mathcal{M})$ predictable satisfying Itô isometry and Burkholder-Davis-Gundy inequality (see also [Barnett-Streater-Wilde,1983]);
- Generalize Itô formula and Girsanov theorem for Itô processes, when H_s (anti)commute with X_s.

The proof of previous results are a combination of the classical ideas of stochastic analysis and the analogous results in Haagerup spaces (see [Pisier–Xu,1997], [Junge,2002], [Junge–Xu,2003], and [Lee–Richard,2011]).

In the paper we propose two applications of stochastic analysis on $\mathbb{L}^p$:

1. A construction "à la Nelson" of the Ψ_2^4 (fermionic) "measure" on $\mathbb{T}^2$;
2. Existence of weak solutions for the stochastic quantization SPDE on $\mathbb{T}^2$ of the Ψ_2^4 (fermionic) model extending the analogous bosonic result of [Jona Lasinio-Mitter,1984].

We consider a Gaussian field defined on $V = (H^{-1+s}(\mathbb{T}^2))^4$, $s \geqslant 0$, of two fermionic fields $\Psi = ((\Psi^1, \bar{\Psi}^1), (\Psi^2, \bar{\Psi}^2))$ with covariance

$$\omega(\Psi^1(x)\bar{\Psi}^1(y)) = \omega(\Psi^2(x)\bar{\Psi}^2(y)) = (-\Delta + m^2)^{-1+s}(x-y)$$

$$\omega(\Psi^i(x)\Psi^i(y)) = \omega(\bar{\Psi}^i(x)\bar{\Psi}^i(y)) = \omega(\Psi^j(x)\bar{\Psi}^i(y)) = 0$$

where $x, y \in \mathbb{T}^2$ and $i, j \in \{1, 2\}$, $i \neq j$. The operators $\Psi = ((\Psi^1, \bar{\Psi}^1), (\Psi^2, \bar{\Psi}^2))$ are not functions, but they can be identified with distributions in the Besov spaces

$$B^{-s-\varepsilon}_{p,p}(\mathbb{T}^2, \mathcal{M}_a)$$

for any $p \geqslant 1$ and $\varepsilon > 0$.

We define the potential

$$V(\Psi) = \int_{\mathbb{T}^2} :\Psi^1(x)\bar{\Psi}^1(x)\Psi^2(x)\bar{\Psi}^2(x): \mathrm{d}x$$

which can be identified with an element of $\mathbb{L}^p_\omega(\mathcal{M})$.

We want to prove the existence of the expectation

$$\omega_V(\cdot) = \frac{\omega(e^{V(\Psi)}\,\cdot\,)}{\omega(e^{V(\Psi)})}.$$

Theorem **(D-Fresta-Gordina-Gubinelli)** *If* $0 \leqslant 7s < 1$, *then* $e^{V(\Psi)}$ *exists and* $e^{V(\Psi)} \in \bigcap_{1 \leqslant p < +\infty} \mathbb{L}^p_\omega(\mathcal{M})$.

Idea of the proof:

Consider a $\chi\colon \mathbb{R} \to [0,1]$ smooth function with compact support and identically one in a neighborhood of 0. Then define

$$((\Psi_t^1, \bar{\Psi}_t^1), (\Psi_t^2, \bar{\Psi}_t^2)) = \Psi_t = \mathcal{F}^{-1}\left(\chi\left(\frac{|\cdot|}{t}\right)\right) * \Psi.$$

We can consider

$$V_t(\Psi_t) = \int_{\mathbb{T}^2} (\Psi_t^1 \bar{\Psi}_t^1 \Psi_t^2 \bar{\Psi}_t^2 - c_t(\Psi_t^1 \bar{\Psi}_t^1 + \Psi_t^2, \bar{\Psi}_t^2) + c_t^2) \mathrm{d}x$$

where $c_t = \sum_{k \in \mathbb{Z}^2} \frac{1}{(|k|^2 + m^2)^{1-s}} \chi\left(\frac{|k|}{t}\right)$.

Lemma *For any* $\nu < 1-(4-1)s = 1-3s$

$$\|V_t(\Psi_t)\|_{\mathbb{L}^\infty} \lesssim t^{4s}, \quad \|T_\tau^{(2)}(V_t(\Psi_t) - V_{t'}(\Psi_{t'}))\|_{L^2_\omega(\mathcal{M})} \lesssim_\tau t'^\nu$$

where the constants $\lesssim_\tau$ *are locally bounded in* $\tau \in \mathbb{R}$.

The constants $c_t \sim t^{2s} \approx \sum_{|k|\leqslant t} \frac{1}{(|k|^2+1)^{1-s}}$ as $t \to +\infty$, furthermore $\|\Psi_t^i\|_{\mathbb{L}^\infty} = \|\bar{\Psi}_t^j\|_{\mathbb{L}^\infty} \sim t^s$.

For the other term we note that

$$\begin{aligned}
\|T_\tau^{(2)}(V_t - V_{t'})\|_2^2 &= \|T_\tau^{(2)}(V_t)\|_2^2 - \|T_\tau^{(2)}(V_{t'})\|_2^2 \\
&= C_\tau \sum_{k_1,\dots,k_4 \in \mathbb{Z}^2} \left(\prod_i \chi(|k_i|/t) - \prod_i \chi(|k_i|/t') \right) \frac{\mathbf{1}_{\sum_i k_i = 0}}{\prod_i (1+k_i^2)^{1-s}} \\
&\lesssim t'^{-\frac{1-2\varepsilon-3s}{1+\varepsilon}}.
\end{aligned}$$

By the previous inequalities and hypercontractivity we get

$$\|V_t\|_{\mathbb{L}^p} \leqslant \|V_{t'}\|_{\mathbb{L}^\infty} + \|V_t - V_{t'}\|_{\mathbb{L}^{p\vee 2}} \lesssim t'^{4s} + p^2\, t'^{-\nu}.$$

In particular, choosing $t' = p^{\frac{2}{4s+\nu}}$ we have $\|V_t\|_{\mathbb{L}^p} \lesssim p^{\frac{8s}{4s+\nu}}$, and by the fact that $7\,s < 1$ implies

$$\|e^{V_t(\Psi_t)}\|_{\mathbb{L}^p} \leqslant \sum_{n\geqslant 0} \frac{\|V_t\|_{\mathbb{L}^{pn}}^n}{n!} \leqslant \sum_{n\geqslant 0} \frac{c^n\,(p\,n)^{\frac{8sn}{4s+\nu}}}{n!}$$

where $\frac{8\,s}{4\,s+\nu} < 1$ since $\nu \sim 1 - 3s$. Thus

$$\|e^{V_t(\Psi_t)} - e^{V_{t'}(\Psi_{t'})}\|_{\mathbb{L}^p} \lesssim \left(\sup_{r=t',t} \|e^{V_{t'}(\Psi_{t'})}\|_{\mathbb{L}^{2p}}\right) \|V_t - V_{t'}\|_{\mathbb{L}^{2p}} \lesssim t'^{-\nu}.$$

Here we consider the approximate Ψ^4_2 equation on $\mathbb{T}^2$

$$\begin{cases} \chi_t^{(N)}(x) = \chi_0^{(N)}(x) - \displaystyle\int_0^t [A^{1-2\theta}\chi_s^{(N)}(x) + \\ \qquad\qquad P_N(A^{-2\theta}[\![\chi_s^{(N)}|\chi_s^{(N)}|^2]\!])(x)]\mathrm{d}s + A^{-\theta}X_t(x), \\ \chi_0^{(N)}(x) = \tilde{X}_0(x) + h_0(x). \end{cases}$$

where $\chi_t^{(N)} = (\chi_t^{(N),1}, \bar{\chi}_t^{(N),1}, \chi_t^{(N),2}, \bar{\chi}_t^{(N),2})$, P_N is the projection on the Fourier modes smaller than $N \in \mathbb{N}$, $A = (-\Delta + m^2)$, $0 \leqslant \theta < 1$, $\tilde{X}_0$ is a Grassmann random field distributed as Ψ in the previous slides, when $s = 0$, and h_0 is a Grassmann random field such that $h_0 \colon \mathbb{T}^2 \to \mathbb{L}^\infty_\omega(\mathcal{M})$ is a C^1 function.

We introduce also the process

$$Z_t^{N,h_0} = \exp\Bigg(\lambda \int_0^t \int_{\mathbb{T}^2} P_N(A^{-\theta} [\![\mathfrak{P}_3(P_N(X_s^A + \mathrm{e}^{-A^{1-2\theta}s} h_0))]\!])(x) \mathrm{d}X_s(x) \\ - \frac{\lambda^2}{2} \int_0^t \int_{\mathbb{T}^2} \langle P_N(A^{-\theta} [\![\mathfrak{P}_3(P_N(X_s^A + \mathrm{e}^{-A^{1-2\theta}s} h_0))]\!]), \rangle_{\mathbb{R}^4} \mathrm{d}x \mathrm{d}s \Bigg), \\ U A^{-\theta} [\![\mathfrak{P}_3(P_N(X_s^A + \mathrm{e}^{-A^{1-2\theta}s} h_0))]\!]$$

where

$$X_t^A = \mathrm{e}^{-A^{1-2\theta}t} \tilde{X}_0 + \int_0^t \mathrm{e}^{-A^{1-2\theta}(t-s)} A^{-\theta} \mathrm{d}X_s,$$

$$\mathfrak{P}_3(P_N(X_s^A + \mathrm{e}^{-A^{1-2\theta}s} h_0)) = P_N(X_s^A + \mathrm{e}^{-A^{1-2\theta}s}\, h_0) |P_N(X_s^A + \mathrm{e}^{-A^{1-2\theta}s}\, h_0)|^2.$$

Using some techniques similar to the ones of the previous slides, we are able to prove that $Z_t^{N,h_0} \in C^0([0,T], \mathbb{L}^p_\omega(\mathcal{M}))$, for every $2 \leqslant p < +\infty$, and that, when θ is close enough to $\frac{1}{2}$, it converges to some $Z_t^{h_0}$, in the same space, as $N \to +\infty$.

Theorem **(D-Fresta-Gordina-Gubinelli)** *Let* $\frac{19}{40} < \theta < \frac{1}{2}$ *and* $h_0 \in C^1(\mathbb{T}^2, \mathbb{L}^\infty)$ *anti-commuting with* X*. Then, for any* $F \in \bigoplus_{n=0}^k \Lambda^n(\mathcal{S}(\mathbb{T}^2)^r)$ *and any* $t_1 < \cdots < t_r \in \mathbb{R}_+$ *we have*

$$\begin{aligned} & \lim_{N \to \infty} \omega_0(F(\chi_{t_1}^{(N)}, \ldots, \chi_{t_r}^{(N)})) \\ = & \lim_{N \to \infty} \omega_0(F(X_{t_1}^A + \mathrm{e}^{-A^{1-2\theta} t_1} h_0, \ldots, X_{t_r}^A + \mathrm{e}^{-A^{1-2\theta} t_1} h_0)\, Z_{t_r}^{h_0,N}) \\ = & \,\omega_0(F(X_{t_1}^A + \mathrm{e}^{-A^{1-2\theta} t_1} h_0, \ldots, X_{t_r}^A + \mathrm{e}^{-A^{1-2\theta} t_1} h_0)\, Z_{t_r}^{h_0}), \end{aligned}$$

where $\chi_t^{(N)}$ *is the solution to approximated* Ψ_2^4 *SPDE.*

Conclusions

We propose a probabilistic theory for the description of fermionic quantum fields, introducing the notion of twisted $\mathbb{L}^p$ spaces. This allows us to develop an anti-commutative stochastic calculus and to apply it to some toy models.

The stochastic quantization program in the case of fermionic systems is only at the beginning, and it needs new developments in particular in the unification of the anti-commutative stochastic calculus and the pathwise analysis of SPDEs.

We think that the improvement of such a theory is useful in constructive field theory, particularly in the case of systems including both fermions and bosons.

Thank you for your attention!

Quantization of bosons

- Simon, Barry. *The $P(\varphi)_2$ Euclidean (quantum) field theory.* Princeton Series in Physics. *Princeton University Press, Princeton, N.J.,* 1974.
- Glimm, James, and Arthur Jaffe. *Quantum physics: a functional integral point of view*. Springer Science & Business Media, 2012.
- Rivasseau, Vincent. *From perturbative to constructive renormalization*. Vol. 46. Princeton University Press, 2014.

Stochastic quantization

- Parisi, Georgio, and Yong Shi Wu. "Perturbation theory without gauge fixing." *Sci. Sin* 24.4 (1981): 483-496.
- Damgaard, Poul H., and Helmuth Hüffel. "Stochastic quantization." *Physics Reports* 152.5-6 (1987): 227-398.

Singular SPDEs

- Hairer, Martin. "A theory of regularity structures." *Inventiones mathematicae* 198.2 (2014): 269-504
- Gubinelli, Massimiliano, Peter Imkeller, and Nicolas Perkowski. "Paracontrolled distributions and singular PDEs." *Forum of Mathematics, Pi*. Vol. 3. Cambridge University Press, 2015
- Kupiainen, Antti. "Renormalization group and stochastic PDEs." *Annales Henri Poincaré*. Vol. 17. No. 3. Springer International Publishing, 2016
- Otto, Felix, and Hendrik Weber. ''Quasilinear SPDEs via rough paths." *Archive for Rational Mechanics and Analysis* 232.2 (2019): 873-950
- Jona-Lasinio, Giovanni, and Pronob K. Mitter. "On the stochastic quantization of field theory." *Communications in Mathematical Physics* 101 (1985): 409-436

Renormalization group applied to fermionic models

- Osterwalder, Konrad, and Robert Schrader. "Euclidean Fermi fields and a Feynman-Kac formula for boson-fermion models." *Helv. Phys. Acta* 46 (1973/74): 277–302.
- Fröhlich, Jürg, and Konrad Osterwalder. "Is there a Euclidean field theory for Fermions?." *Helv. Phys. Acta* 47 (1974/75): 781–805.
- Gawçdzki, Krysztof, and Antti Kupiainen. "Gross-Neveu model through convergent perturbation expansions." *Communications in mathematical physics* 102.1 (1985): 1-30
- Feldman, J., J. Magnen, V. Rivasseau, and R. Sénéor. "A renormalizable field theory: the massive Gross-Neveu model in two dimensions." *Communications in mathematical physics* 103 (1986): 67-103.
- Lesniewski, A. "Effective action for the Yukawa_2 quantum field theory." *Commun. Math. Phys.;* 108.3 (1987)
- Benfatto, G., and G. Gallavotti. "Renormalization-group approach to the theory of the Fermi surface." *Physical Review B* 42.16 (1990): 9967.
- Disertori, Margherita, and Vincent Rivasseau. "Continuous constructive fermionic renormalization." *Annales Henri Poincaré*. Vol. 1. No. 1. Birkhäuser Verlag, 2000
- Giuliani, Alessandro, Vieri Mastropietro, and Marcello Porta. "Universality of the Hall conductivity in interacting electron systems." *Communications in Mathematical Physics* 349.3 (2017): 1107-1161
- Giuliani, Alessandro, Vieri Mastropietro, and Slava Rychkov. "Gentle introduction to rigorous Renormalization Group: a worked fermionic example." *Journal of High Energy Physics* 2021.1 (2021);
- Salmhofer, Manfred. *Renormalization: an introduction*. Springer Science & Business Media, 2007
- Mastropietro, Vieri. *Non-perturbative renormalization*. World Scientific, 2008

Segal gage spaces

- Segal, Irving E. "A non-commutative extension of abstract integration." *Annals of mathematics* (1953): 401-457
- Carlen, Eric A., and Elliott H. Lieb. "Optimal hypercontractivity for Fermi fields and related non-commutative integration inequalities." *Communications in mathematical physics* 155.1 (1993): 27-46
- Barnett, C., R. F. Streater, and I. F. Wilde. "The ito-clifford integral." *Journal of Functional Analysis* 48.2 (1982): 172-21.

Tomita–Takesaki theory

- Haagerup, Uffe. "Lp-spaces associated with an arbitrary von Neumann algebra." *Algebres d'opérateurs et leurs applications en physique mathématique (Proc. Colloq., Marseille, 1977)*. Vol. 274. 1979.
- Pisier, Gilles, and Quanhua Xu. "Non-commutative martingale inequalities." *Communications in mathematical physics* 189 (1997): 667-698
- Junge, Marius. "Doob's inequality for non-commutative martingales". *J.ReineAngew.Math.*, 549:149–190, (2002)
- Junge, Marius, and Quanhua Xu. "Noncommutative Burkholder/Rosenthal inequalities." *The Annals of Probability* 31.2 (2003): 948-995
- Lee, Hun Hee, and Éric Ricard. "Hypercontractivity on the q-Araki-Woods algebras." *Communications in mathematical physics* 305.2 (2011): 533-553
- Majewski, Adam W., and Boguslaw Zegarlinski. "On quantum stochastic dynamics and noncommutative spaces." *Letters in Mathematical Physics* 36.4 (1996): 337-349.
- Araki, Huzihiro, Walter Wyss. "Representations of canonical anticommutation relations." *Helv. Phys. Acta* 37 (1964): 136–159.
- Barnett, C., R. F. Streater, and I.F. Wilde. "Quasi-free quantum stochastic integrals for the CAR and CCR." *Journal of Functional Analysis* 52.1 (1983): 19-47

Stochastic quantization/stochastic analysis for fermions

- Albeverio, Sergio, Luigi Borasi, Francesco C. De Vecchi and Massimiliano Gubinelli. "Grassmannian stochastic analysis and the stochastic quantization of Euclidean Fermions." *Probability Theory and Related Fields* (2022): 1-87
- De Vecchi, Francesco C., Luca Fresta, and Massimiliano Gubinelli. "A stochastic analysis of subcritical Euclidean fermionic field theories." *arXiv preprint arXiv:2210.15047* (2022)
- Chandra, Ajay, Martin Hairer, and Martin Peev. "A Dynamical Yukawa$_2$ Model." *arXiv preprint arXiv:2305.07388* (2023)
- De Vecchi, Francesco C., Luca Fresta, Maria Gordina, and Massimiliano Gubinelli. "Non-commutative L^p spaces and Grassmann stochastic analysis." *arXiv preprint arXiv:2305.08497* (2023).