# Measured asymptotic expanders and rigidity of Roe algebras

(Joint with K Li and J Zhang)

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# Outline

1. Roe algebras, and the result

2. (Measured, asymptotic) expanders

# Coarse Geometry

”Study metric spaces from a large–scale perspective” (Gromov, Roe).

**Definition**

An $f : X \to Y$ is a *coarse embedding*, if there exist $\rho_+, \rho_- : \mathbb{R}_+ \to \mathbb{R}_+$, $\rho_- \nearrow \infty$, with $\rho_-(d(x,y)) \leq d(f(x),f(y)) \leq \rho_+(d(x,y))$ for $x, y \in X$.
A c. emb. $f$ is a *coarse equivalence* ($\sim_c$), if $\sup_{y \in Y} d(y,f(X)) < \infty$.

**Why?** Theorems in geometry ($\nexists$ PSC metrics), topology (Novikov conjecture) via functional analysis (Roe algebras).

# Hitchhiker's guide to coarse properties

Finite asymptotic dimension

$\Downarrow$

Finite decomposition complexity (FDC)

$\Downarrow$

Property A (aka MSP, ONL, …)

$\Downarrow$

Coarse embeddability into $\ell^2$

$\Downarrow$

Coarse embeddability into $\ell^p$

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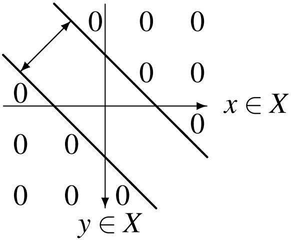
Here be dragons
(mostly expanders)

# ROE ALGEBRAS

Let $(X, d)$ be a discrete metric space with bounded geometry.
(bdd.geom. means $\forall R \geq 0$: $\sup_{x \in X} |B(x, R)| < \infty$.)

**Roe algebras** are C*- (or Banach-) algebras which encode the coarse geometry of $X$.




An operator $T = (T_{yx})_{x,y \in X}$ on $\ell^2 X$ has **finite propagation** (or a **band operator**) if $\exists R \geq 0$, such that $d(x, y) \geq R$ implies $T_{yx} = 0$.

These form a $*$-algebra in $\mathscr{B}(\ell^2 X)$. Its norm closure is called $C^*_u X$, the **uniform Roe algebra** of $X$.
[ Can also do $\ell^p X$ for $p \in [1, \infty]$. ]

So the elements of $C^*_u X$ are norm-limits of finite propagation operators; a.k.a. *band-dominated operators*.

**Proposition**: $X \sim_c Y \implies C^*_u X$ and $C^*_u Y$ are (stably) isomorphic.

# Rigidity of Roe algebras

Results of the sort "$C_u^*X \simeq C_u^*Y \implies X \simeq Y$".
...for the appropriate versions of "$\simeq$".
Usually require some assumption on $X$ and/or $Y$.

For example: [S-Willett '10]
"$C_u^*X \otimes \mathscr{K} \cong C_u^*Y \otimes \mathscr{K}$ and $X$ has Property A $\implies X \sim_c Y$"

Many more results now; work of *Braga, Farah*, Vignati, Chung, Li,...

- stronger conclusions (e.g. bijective c.eq.)
- more general $X$ (coarse spaces)
- "embeddings" versions: replace $\simeq$ by $\to$
- weaker assumptions (e.g. coarse embeddability into $\ell^2$; or a 'technical condition')

# A rigidity result

Let $X$ and $Y$ be metric spaces of bounded geometry.

### Theorem (Li-S-Zhang)

*If $X$ contains no sparse subspaces which form a ghostly measured asymptotic expander, then $C_u^*X \otimes \mathscr{K} \cong C_u^*Y \otimes \mathscr{K} \implies X \sim_c Y$.*

($Y \subseteq X$ is *sparse*, if $Y = \sqcup_n Y_n$, each $Y_n$ is finite, and $d(Y_m, Y_n) \to \infty$.)
($X = \sqcup_n G_n$ is an **e.**an **a.e.**a **m.a.e.** if $\exists$ prob. measures $m_n$ on $G_n$ s.t. $\forall \alpha \in (0, \frac{1}{2}]$:

$$\inf_n \inf \left\{ \frac{|\partial A|}{|A|} \mid A \subset G_n, 0 < |A| \leq \frac{|G_n|}{2} \right\} > 0.$$

$$\inf_n \inf \left\{ \frac{|\partial A|}{|A|} \mid A \subset G_n, \alpha|A| < |A| \leq \frac{|G_n|}{2} \right\} > 0.$$

$$\inf_n \inf \left\{ \frac{m_n(\partial A)}{m_n(A)} \mid A \subset G_n, \alpha m_n(G_n) < m_n(A) \leq \frac{m_n(G_n)}{2} \right\} > 0.$$

It is **ghostly** if and $\lim_n \sup_{x \in G_n} m_n(x) = 0$.)

### Corollary (L-S-Z)

*Assume that $X$ coarsely embed into $\ell^p$, $p \in [1, \infty)$. Same conclusion.*

### Theorem (L-S-Z)

# QUASI-LOCALITY

$T \in \mathscr{B}(\ell^2 X)$ has finite propagation $\iff$
$\exists R \geq 0\ \forall A, B \subset X$ with $d(A,B) \geq R$ we have $\chi_A T \chi_B = 0$.

If $T \in C_u^* X$, then $\forall \varepsilon > 0$
$\exists R \geq 0\ \forall A, B \subset X$ with $d(A,B) \geq R$ we have $\|\chi_A T \chi_B\| < \varepsilon$.

We say that $T \in \mathscr{B}(\ell^2 X)$ is **quasi-local**, if $\forall \varepsilon > 0$ ... (that condition ↑).

**Question:** Does every quasi-local operator on $\ell^2 X$ belong to $C_u^* X$?

A remark in J Roe's CBMS 1996 book. Answers: yes, if...

- $X = \mathbb{Z}^n$: Rabinovich–Roch–Silbermann (uses Fourier transform).
- $X$ has polynomial growth: A Engel, 2016
- $X$ has straight FDC; Tikuisis–S, 2016
- $X$ has Property A; S–Zhang, 2018

How about **a counterexample** (i.e. a quasi-local operator not in the Roe algebra)?

# Expanders

- For a finite graph $G$, the **graph Laplacian** is $\Delta_G \in \mathscr{B}(\ell^2 G)$ given by $\Delta_G \xi(x) = \xi(x) - \sum_{y\sim x} \frac{1}{\sqrt{\deg(x)\deg(y)}} \xi(y)$.
- $\Delta_G$ has propagation 1.
- $\Delta_G \geq 0$; denote $\lambda_G$ the smallest non-zero eigenvalue of $\Delta_G$.
- $G$ connected $\implies$ 0-eigenspace is spanned by $f(x) = \sqrt{\deg(x)}$.
- Denote $P_G \in \mathscr{B}(\ell^2 G)$ the 0-spectral projection.

A sequence $(G_n)_n$ of (connected) finite graphs is an **expander**, if

- $\sup\{d_x \mid n \in \mathbb{N}, x \in G_n\} < \infty; \quad |G_n| \to \infty$
- $\inf_n \lambda_{G_n} > 0$.

  $\iff \inf_n \inf \left\{ \frac{|\partial A|}{|A|} \mid A \subset G_n, 0 < |A| \leq \frac{|G_n|}{2} \right\} > 0$

  ("Cheeger constants" uniformly away from 0)

Denote $X = \sqcup_n G_n$. Then $\oplus_n \Delta_{G_n} \in C^*_u X$.
If $(G_n)_n$ is an expander, then (spectral gap $\implies$) $P = \oplus_n P_{G_n} \in C^*_u X$ .
This $P$ is a "ghost projection" (its matrix entries go to zero, but it is not compact). [Counterexample to coarse Baum-Connes conjecture.]

# ASYMPTOTIC EXPANDERS

Find a space $X = \sqcup G_n$ where $P$ is quasi-local, but not in $C^*_u X$?

But... **no**. $P$ is quasi-local
$\iff$ $X$ is an **asymptotic expander** [Nowak-Li-S-Zhang]

$$\forall \alpha \in (0, \tfrac{1}{2}]\colon \inf\left\{ \frac{|\partial A|}{|A|} \mid A \subset G_n, \alpha|G_n| < |A| \leq \frac{|G_n|}{2}, n \in \mathbb{N} \right\} > 0$$

$\iff$ $P \in C^*_u X$ [Khukhro-Li-Vigolo-Zhang]

**Thm:** [KLVZ] $X = \sqcup_n G_n$ is an asymptotic expander if and only if it admits "uniform exhaustion by expanders":

There exist sequences $\{\alpha_k > 0\}_k$, $\{c_k > 0\}_k$, such that $\alpha_k \to 0$, and subsets $Y_{n,k} \subseteq X_n$ such that $\{Y_{n,k}\}_n$ is a $c_k$-expander and $|Y_{n,k}| \geq (1 - \alpha_k)|X_n|$ for every $n, k$.

( $\implies$ If $X$ is an a.e., then $P$ is approximated in norm by averaging projections on the expanders that exhaust $X$.)

[KLVZ] also prove: If an asymptotic expander is "symmetric" (e.g. the graphs are vertex-transitive), then it is an expander.

# Measured version

How about $S = \oplus S_n \in \mathscr{B}(\ell^2 X)$ where $S_n \in \mathscr{B}(\ell^2 G_n)$ is a rank one projection? (A "block-rank-one" projection.)

Still **no**. $S$ is quasi-local
$\iff$ $X$ is a **measured asymptotic expander**:
$\exists$ probability measures $m_n$ on $G_n$ so that $\forall \alpha \in (0, \frac{1}{2}]$:
$\inf_n \inf \left\{ \frac{m_n(\partial A)}{m_n(A)} \mid A \subset G_n, \alpha m_n(G_n) < m_n(A) \leq \frac{m_n(G_n)}{2} \right\} > 0$
$\iff$ $S \in C_u^* X$ [Li-S-Zhang]

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Formulas: $S = \oplus S_n \quad \rightsquigarrow \quad S_n = \langle . | \xi_n \rangle \xi_n \quad \rightsquigarrow \quad m_n(x) = |\xi_n(x)|^2$
Also: $\|\chi_A S_n \chi_B\| = \|\chi_A \xi_n\| \|\chi_B \xi_n\| = \sqrt{m_n(A) m_n(B)}$
Thus: $S$ quasi-local $\iff$
$0 = \lim_{R\to\infty} \sup\{m_n(A) m_n(B) : n \in \mathbb{N},\ A, B \subseteq X_n,\ d(A,B) \geq R\}$

# Measured asymptotic expanders

**Def:** $X = \sqcup_n G_n$ is a **measured asymptotic expander** if there exist probability measures $m_n$ on $G_n$ so that $\forall \alpha \in (0, \frac{1}{2}]$:

$$\inf_n \inf \left\{ \frac{m_n(\partial A)}{m_n(A)} \mid A \subset G_n, \alpha m_n(G_n) < m_n(A) \leq \frac{m_n(G_n)}{2} \right\} > 0.$$

It is **ghostly** if and $\lim_n \sup_{x \in G_n} m_n(x) = 0$.

- ... don't embed into any $L^p$-space, $p \in [1, \infty)$.
- ... examples? hm....
- ... satisfy a Structure Theorem (like asymptotic expanders), with:

**Def:** A sequence $(G_n, m_n)$ of measured graphs ($m_n$: full measures) is a **measured expander**, if:

$$\inf_n \inf \left\{ \frac{m_n(\partial A)}{m_n(A)} \mid A \subset G_n, 0 < m_n(A) \leq \frac{m_n(G_n)}{2} \right\} > 0.$$

# A little about Measured Expanders

**Def:** The *Cheeger constant* of a measured graph $(G,m)$ is
$ch(G,m) = \inf\left\{ \frac{m(\partial A)}{m(A)} \mid A \subset G, 0 < m(A) \leq \frac{m(G)}{2} \right\}$.

**Def:** A sequence $(G_n, m_n)$ of measured graphs ($m_n$: full measures) is a **measured expander**, if: $\inf_n ch(G_n, m_n) > 0$.

- We don't have a nice positive graph Laplacian...
- ... unless the measures come from random walks (i.e. weights on edges).
- So to do anything, we need an extra assumptions: "bounded measure ratio" (the quantity $m(x)/m(y)$ is bounded from below and above by some $s > 0$, for every edge $x \sim y$) and bounded valency.
- With this, we can get Poincarè inequalities:

$$\sum_{x \sim y} |f(x) - f(y)|^p (m(x) + m(y)) \geq c_p \sum_{x,y} |f(x) - f(y)|^p \frac{m(x)m(y)}{m(G)}.$$

- Examples come from actions on prob. measure spaces with a spectral gap.

# AN EXAMPLE OF A MEASURED EXPANDER
## … WHICH IS NOT AN EXPANDER (BY GÁBOR ELEK)

Let $G$ be a graph. Endow it with the normalised counting measure $\mu$.
Endow the graph $G^\infty := G \times \mathbb{N}_0$ with the measure $\mu^\infty$ which restricts to $2^{-k}$ on $G \times \{k\}$. (Then $\mu^\infty(G^\infty) = 2$.)

*Exercise:* If the Cheeger constant of $G$ is at least $c > 0$, then the Cheeger constant of $G^\infty$ is at least $\min(c/18, 1/8)$.

Let $(G_n)_n$ be an expander, endowed with normalised counting measures $m_n$.
Let $X_n = G_n \times \{0, \dots, n\}$, with the measure $m_n$ which restricts from $G_n^\infty$.

Then $(X_n)_n$ is a measured expander, which is not an expander.

# A rigidity result (again)

Let $X$ and $Y$ be metric spaces of bounded geometry.

**Theorem (Li-S-Zhang)**

*If X contains no sparse subspaces which form a ghostly measured asymptotic expander, then* $C_u^*X \otimes \mathscr{K} \cong C_u^*Y \otimes \mathscr{K} \implies X \sim_c Y$.

**Corollary (L-S-Z)**

*Assume that X coarsely embeds into* $\ell^p$, $p \in [1, \infty)$. *Same conclusion.*

**Theorem (L-S-Z)**

*The spaces constructed by Arzhantseva–Tessera ('12) and Delabie–Khukhro – which do not contain expanders, nor coarsely embed into any* $\ell^p$ *– satisfy the assumption of the Theorem above.*

# ”SOME DETAILS”

About the proof: Essentially just a small improvement on existing proofs of B-F-V, B-C-L:
Their ”technical condition” says ”no sparse subspaces yield non-compact ghost projections”.

We improve to ”no sparse subspaces yield non-compact block-rank-one ghost projections”.

Then show that ”containing a gmae” implies there is a sparse subspace with a ghost block-rank-one projection.

# A CONNECTION TO THE COARSE BAUM–CONNES CONJECTURE

(somewhat tentative ...)

**THEOREM**

*Let $X = \sqcup_n X_n$. If X admits a fibred coarse embedding into a Hilbert space, and X "coarsely contains" a gmae, then:*

(1) *The coarse Baum–Connes assembly map $\mu$ for X is injective but non-surjective.*
(2) $\iota_* : K_*(\mathscr{K}) \to K_*(I_G)$ *is injective but non-surjective, where $I_G$ is the ghost ideal $I_G \lhd C^*(X)$.*
(3) $\pi_* : K_*(C^*_{max}(X)) \to K_*(C^*(X))$ *is injective but non-surjective.*

# QUESTIONS...

- More examples...
- Rigidity of Roe algebras for (measured?) expanders?
- Quasi-local operators that are not in Roe algebras?

Thank you!