Median Geometry for Lattices

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9 June 2021, Virtual Münster

Some quotations

John von Neumann: “If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.”

Guillermo Moreno: “Groups, as men, will be known by their actions.”

Bernt Øksendal: “We have not succeeded in answering all our problems. The answers we have found only serve to raise a whole set of new questions. In some ways we feel we are as confused as ever, but we believe we are confused on a higher level and about more important things.”

Median Geometry

In this talk I shall explain how various types of lattices of isometries of (products of) hyperbolic spaces present various degrees of compatibility with the median geometry and its discrete version, the CAT(0) cubical complex geometry.

Strongest degree of compatibility with median geometry

A group is said to be cubulable if it acts properly discontinuously cocompactly on a CAT(0) cube complex.

Theorem (Bergeron-Wise, Kahn-Markovic)

Every fundamental group of a hyperbolic 3*-manifold is cubulable.*

Using this and the theory of special cube complexes (Haglund-Wise), Agol proved the virtual Haken conjecture.

Why is the geometry of a CAT(0) cube complex median

Chepoi, Guerasimov: a graph $\Gamma = (V, E)$ is the 1-skeleton of a CAT(0) cube complex $\Leftrightarrow$ V with the simplicial distance is median.

A median space = a metric space (X, d) such that every triple of points $x_1, x_2, x_3 \in X$ admits a unique median point $m \in X$ satisfying

$$d(x_i, m) + d(m, x_j) = d(x_i, x_j)$$

for all $i, j \in \{1, 2, 3\}, i \neq j$.

Comparison with expander graphs:

- While expander graphs represent robust networks, difficult to disconnect (i.e. to disconnect a set A of vertices one must remove approx. $c|A|$ edges), difficult to embed into Hilbert spaces (and presumably any reasonable Banach),
- median graphs are economic networks (least number of edges used to connect all the vertices), relatively easy to disconnect, easy to embed into L^1, hence Hilbert spaces.

Median spaces and L^1–spaces

Besides the set of vertices of a CAT(0) cube complex, with the simplicial metric, other examples of median metric spaces are:

Other examples

1. *Real trees;*
2. $\mathbb{R}^n$ *with the norm* ℓ^1*;*
3. $L^1(X, \mu)$.

- (Assouad) Every median space embeds isometrically into an L^1–space.
- (Verheul) A complete median normed spaces is linearly isometric to an L^1–space.

Median spaces: non-discrete versions of CAT(0)-c.c.

Median spaces = non-discrete versions of 0-skeleta of CAT(0) cube complexes.
Analogous to real trees = non-discrete versions of simplicial trees.
Bowditch: The metric of a complete connected finite rank median metric space has a bi-Lipschitz equivariant deformation that is CAT(0) and has the same collection of convex subsets.

The rank of a median metric space X = the supremum over the set of integers k such that X contains an isometric copy of the set of vertices $\{-a, a\}^k$ of the cube of edge length $2a$, for some $a > 0$.

Convention

From now on we assume all median spaces to be complete and connected (hence geodesic).

Interest of the median geometry

1. G locally compact second countable has property (T) $\Leftrightarrow$ any continuous action by isometries on a median space has bounded orbits (Chatterji -D. - Haglund).
2. G a-(T)-menable $\Leftrightarrow$ it admits a proper continuous action by isometries on a median space (Chatterji -D. - Haglund).
3. certain groups and spaces have median geometry asymptotically. So median geometry appears when compactifying spaces of representations.
    1. mapping class groups of surfaces, with word metrics (Behrstock - D. - Sapir);
    2. Teichmüller spaces with Weil-Petersson metric (Bowditch).
4. relevance of median graphs and their geometry in graph theory, computer science, optimization theory.

Degrees of compatibility with median geometry

A group is said to be

- cubulable if it acts properly discontinuously cocompactly on a CAT(0) cubical complex;
- strongly medianizable if it acts properly discontinuously cocompactly on a median space of finite rank;
- (weakly) medianizable if it acts properly discontinuously coboundedly on a median space.

If a finitely generated group is strongly medianizable then the median space on which it acts is also locally compact.

Cubulable $\Rightarrow$ strongly medianizable $\Rightarrow$ (weakly) medianizable.

Degrees of median compatibility versus degrees of amenability

Extra properties of cubulable groups: they are weakly amenable with Cowling-Haagerup constant 1 (Guentner-Higson).

- Amenable$=$ $\exists$ a sequence of positive definite, compactly supported functions on G converging to 1 uniformly on compacts subsets;
- a-T-menable $=$ $\exists$ a sequence of continuous positive definite functions on G, vanishing at infinity, converging to 1 uniformly on compact sets.
- weakly amenable$=$ $\exists$ a sequence (ϕ_n) of continuous, compactly supported functions on G, converging to 1 uniformly on compact sets, with $\sup_n \|\phi_n\|_{M_0A(G)} < \infty$.
  Above, $\|\cdot\|_{M_0A(G)}$ is the completely bounded norm on the space $M_0A(G)$ of completely bounded multipliers of the Fourier algebra $A(G)$.
  The Cowling-Haagerup constant is the best possible $\kappa \geq \sup_n \|\phi_n\|_{M_0A(G)}$.

Strongly medianizable versus cubulable

Possibly strongly medianizable $\Rightarrow$ cubulable.

They share the same properties:

- Tits alternative, super-rigidity (Caprace-Sageev, Fioravanti);
- irreducible uniform lattices in $SO(n_1, 1) \times \cdots \times SO(n_k, 1)$ with $k \geq 2$ act on every finite dimensional CAT(0)-c.c. with a global fixed point (Chatterji-Fernos-Iozzi)
- also true for actions on finite rank median spaces (Fioravanti).

Theorem

(Chatterji-D.) The above lattices (with $k \geq 1$) are medianizable.

In view of Fioravanti's result, this is the best that one can get for these lattices, in terms of compatibility with a median geometry.

It is unknown if the same lattices can act properly on an infinite dimensional CAT(0) cube complex.

Medianizable lattices

The result is interesting in the case of one factor ($k = 1$) too:

- not known if all arithmetic uniform lattices in $SO(n, 1)$, with n odd and larger than 3, are cubulable;
- in particular, there is an example of arithmetic uniform lattices in $SO(7, 1)$, constructed using octaves, that is thought not to be cubulable.

Rips-type Theorems

When can one extract from an action on a real tree $T \neq \mathbb{R}$ (minimal non-trivial) an action on a simplicial tree ?

Stable action = the family of stabilizers of non-trivial arcs satisfies the ACC (Ascending Chain Condition).

- (Bestvina-Feighn) If G is finitely presented then there exists an action on a simplicial tree with stabilizers of edges = stabilizers of arcs-by-cyclic.
- (Sela) Same with f.p. replaced by "trivial stabilizers of tripods".

Rips-type Theorems for median spaces

Interest of a median version:

- it would relate the negation of property (T) ($\Leftrightarrow$ existence of an action with infinite orbits on a median space) to actions on CAT(0) cube complexes;
- it would provide an assumption under which a-T-menability implies weak amenability with Cowling-Haagerup constant 1.
- An assumption is needed: $H \wr F_2$ with H finite is a-T-menable (Cornulier-Stadler-Valette), but cannot be weakly amenable with Cowling-Haagerup constant 1 (Ozawa-Popa).

Rips-type Theorems for median spaces

Our theorem emphasizes that one cannot expect, for actions on median spaces, a theorem similar to Bestvina-Feighn:

- uniform lattices in $SO(n_1, 1) \times \cdots \times SO(n_k, 1)$ are finitely presented;
- they act properly, minimally and with bounded quotient on median spaces;
- they cannot act non-trivially cocompactly with amenable stabilizers on a CAT(0) cube complex (which would therefore have to be of finite dimension), by Chatterji-Fernos-Iozzi.

Rips-type Theorems for median spaces

- Still possible to obtain Rips-type theorems for actions on median spaces of finite rank.
- consistent with the case of real trees, since these are median spaces of rank one;
- a good candidate for the condition "tree not a line" might be median space with no global fixed point at infinity under the full isometry group, not within bounded Hausdorff distance from a space $\mathbb{R}^n$ with ℓ^1 norm"

Where does the median geometry come from ?

Theorem

The real hyperbolic space $\mathbb{H}^n$ embeds isometrically and $\mathrm{Isom}(\mathbb{H}^n)$–equivariantly into a median space at finite Hausdorff distance from the embedded $\mathbb{H}^n$.

The embedding is constructed using another structure closely connected with the median geometry: measured walls (introduced by Cherix-Martin-Valette).

Space with measured walls

A space with measured walls is:

- a space X
- with $\mathcal{W}$ a collection of walls, i.e. pairs $w = \{h, h^c\}$
- $\mathcal{B}$ a σ-algebra of subsets in $\mathcal{W}$
- μ a measure on $\mathcal{B}$, such that for every two points $x, y \in X$ the set of separating walls $\mathcal{W}(x|y)$ is in $\mathcal{B}$ and it has finite measure.

Denote by pdist_μ the pseudo-metric on X defined by

$$\mathrm{pdist}_\mu(x, y) = \mu\left(\mathcal{W}(x|y)\right).$$

We call it the wall pseudo-metric.

Three key features of spaces with measured walls

Chatterji-Drutu-Haglund:

- the category of spaces with measured walls is equivalent to the category of subspaces of pseudo-metric median spaces;
- a median metric space is endowed with a structure of measured walls that are convex (i.e. every h and h^c are convex), such that the wall pseudo-metric pdist_μ coincides with the median metric, and isometries are automorphisms of the space with measured walls;
- to a space with measured walls $(X, \mathcal{W}, \mathcal{B}, \mu)$ one can associate a median pseudo-metric space $(\mathcal{M}(X), \mathrm{pdist})$ containing it, such that the wall pseudo-metric pdist_μ coincides with the restriction of pdist to $X \times X$; moreover $\mathcal{M}(X)$ is minimal, and an isomorphism of $(X, \mathcal{W}, \mathcal{B}, \mu)$ induces an isometry of $(\mathcal{M}(X), \mathrm{pdist})$.

Generalized version of the embedding theorem of $\mathbb{H}^n$

Theorem (Chatterji-D.)

Let $(X, \mathcal{W}, \mathcal{B}, \mu)$ be a space with measured walls, μ-locally finite. The following are equivalent:

1. *the Hausdorff distance from (the embedded copy of) X to $\mathcal{M}(X)$ is finite;*
2. *there exists a median space $\mathcal{M}$ and a coarsely surjective monomorphism $\varphi : X \to \mathcal{M}$;*
3. *there exists δ and D such that X with the wall pseudo-metric is δ-tripodal, and every point z that is δ–between two points x, y is in the D-neighborhood of every half-space containing x and y.*

Complex hyperbolic space

- $(\mathbb{H}^n_{\mathbb{C}}, \mathrm{dist})$ cannot be isometrically embedded into a median space. In particular, dist cannot be a wall metric.
- Faraut and Harzallah: $\mathbb{H}^n_{\mathbb{C}}$ equipped with $\mathrm{dist}^{\frac{1}{2}}$ has a structure of measured walls.
- For any $\alpha \in [1/2, 1)$, whenever dist^{α} is a wall metric, $(\mathbb{H}^n_{\mathbb{C}}, \mathrm{dist}^{\alpha})$ cannot be at bounded Hausdorff distance from a median space.