Noise, differential equations and quantum fields

Mid-Term Conference

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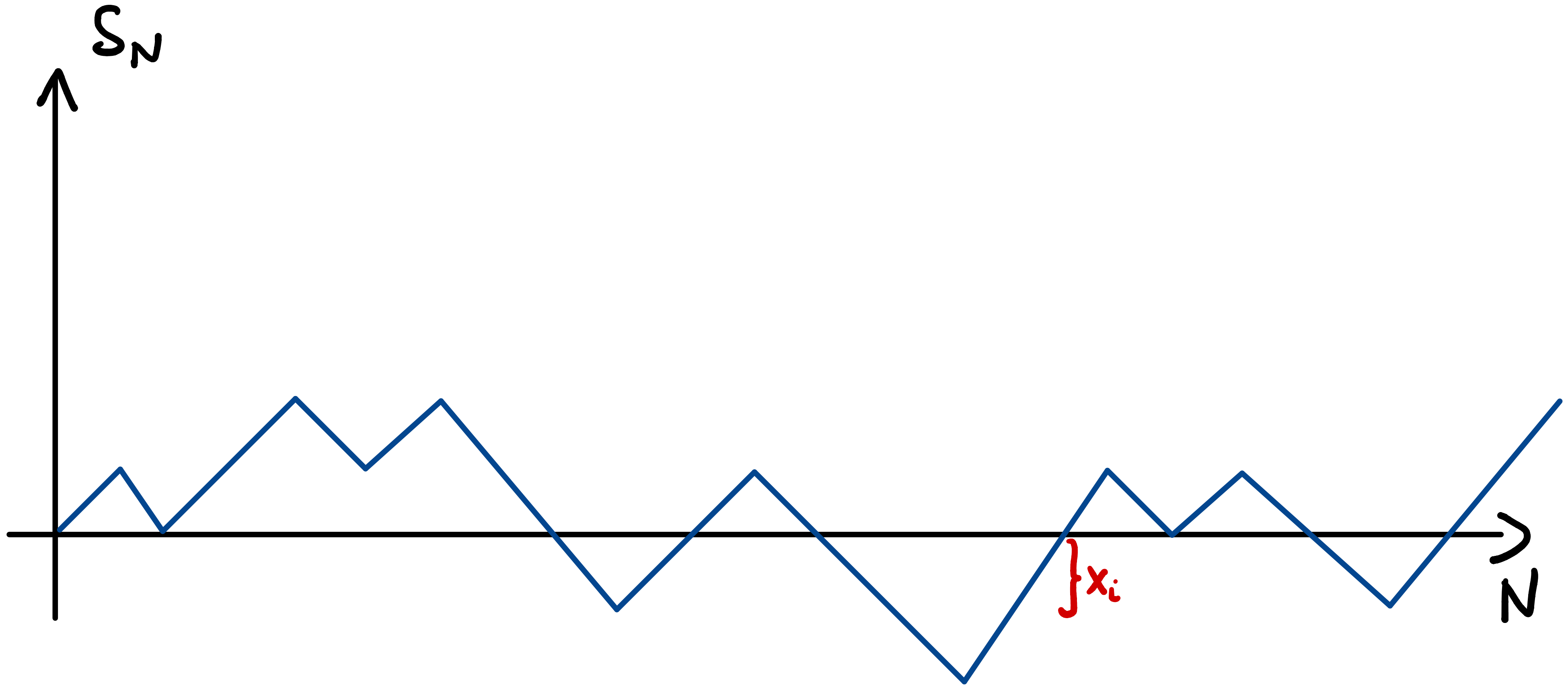
Noise: A classical result on Random walks

Random walk $\quad X_i = \begin{cases} 1 & \text{if head} \\ -1 & \text{if taie} \end{cases} \qquad S_N = \sum_{i=1}^{N} X_i$



Donsker's invariance principle

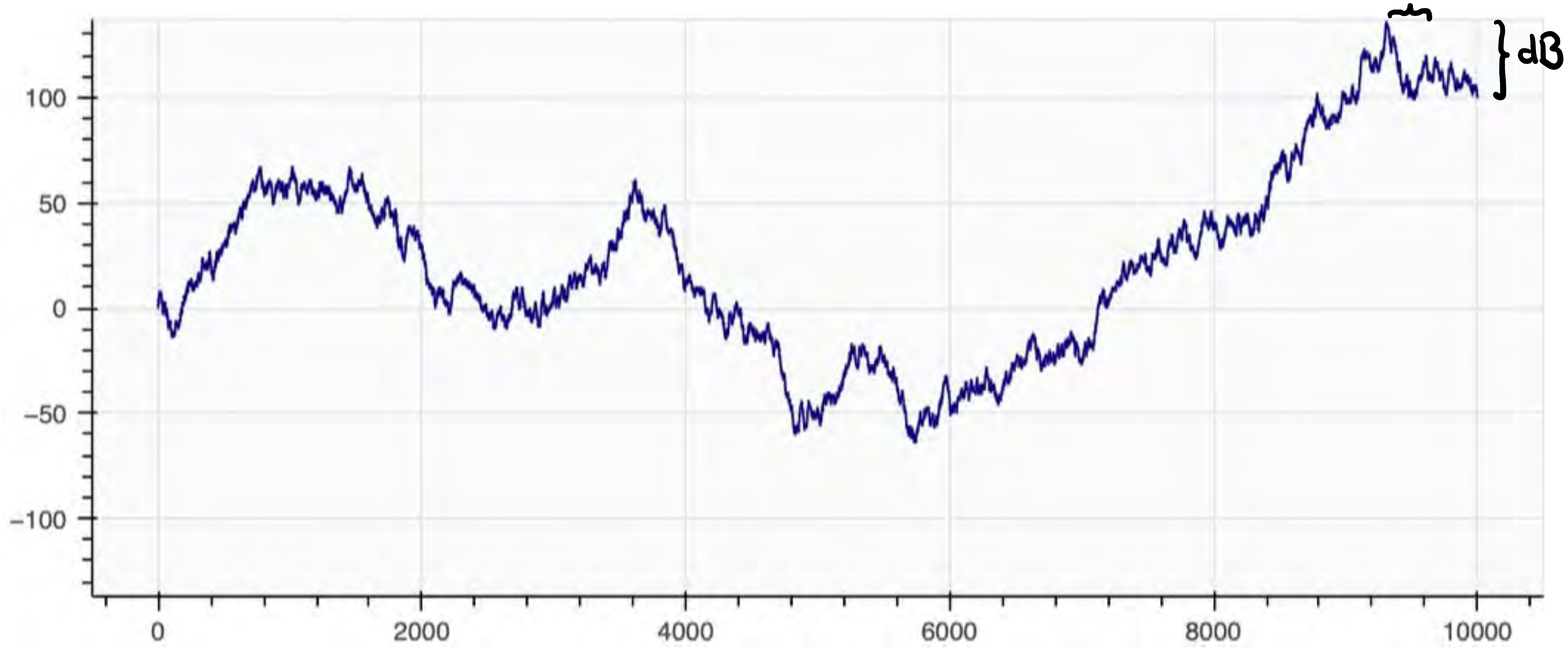
$B_t^N = \frac{1}{\sqrt{N}} S_{Nt}$ ← (piecewise linear interpolation)

Then $B_t^N \Rightarrow B_t$ Brownian motion

"Statistics of path functionals

e.g. $\sup_{t\in[0,1]} B_t$ converge.

Brownian motion



View Brownian motion as "integrated noise" $\boxed{B_t = \int_0^t \xi(s)\,ds}$ white noise.

- In other words "white noise ξ = (distributional) derivative of B_t"
- Formally: $\boxed{\mathbb{E}\,\xi(t)\,\xi(s) = \delta(t-s)}$ Dirac δ-Function

Rigorous interpretation — no problem! but very irregular "½ times diff'able"

"Definition: Stochastic analysis = analysis with white noise"

Itô's stochastic differential equations

noise

$$\partial_t X(t) = b(X(t)) + \sigma(X(t))\,\xi(t)$$

a.k.a. $dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t$

drift diffusion coefficient

Kiyosi Itô 1954

Quelle: Wikimedia.org

Problem: Nonlinear operation on ξ $\sigma(X(t))\,\xi(t)$

Itô calculus (1940s) Solved based on "adaptedness", flow of information. If (Y_s) adapted then

$$\boxed{\mathbb{E}\int_0^t Y_s\,dB_s = 0} \quad \text{and} \quad \boxed{\mathbb{E}\left(\int_0^t Y_s\,dB_s\right)^2 = \mathbb{E}\int_0^t Y_s^2\,ds}$$

Itô isometry

An example (T. Lyons '91)

White noise on $[0,1]$: (a_i), (b_i) i.i.d. $\sim \mathcal{N}(0,1)$

$$c_n(x) = \begin{cases} 1 & n = 0 \\ \sqrt{2}\cos(2\pi n x) & n \geq 1 \end{cases}$$

$$s_n(x) = \sqrt{2}\sin(2\pi n x)$$

$$\boxed{\xi = a_0 c_0 + \sum_{n=1}^{\infty} (a_n c_n + b_n s_n)}$$

Brownian motion as $\int_0^t \xi$

uniform convergence Wiener 1923

$$\boxed{B(t) = a_0 t + g(t) - g(0) \quad \text{where} \quad g(t) = \sum_{n=1}^{\infty} \frac{1}{2\pi n}\left(a_n s_n(t) - b_n c_n(t)\right)}$$

Define

$$\boxed{\tilde{B}(t) = a_0 t + \tilde{g}(t) - \tilde{g}(0) \quad \text{where} \quad \tilde{g}(t) = \sum_{n=1}^{\infty} \frac{1}{2\pi n}\left(a_n c_n(t) + b_n s_n(t)\right)}$$

$B(t)$ and $\tilde{B}(t)$ are both Brownian motions but <u>not</u> independent.

Question: Define $\boxed{\int_0^1 \tilde{B}_t \, dB_t}$

$$\boxed{\begin{aligned} g(t) &= \sum_{n=1}^{\infty} \frac{1}{2\pi n} \left(a_n s_n(t) - b_n c_n(t) \right) \\ \tilde{g}(t) &= \sum_{n=1}^{\infty} \frac{1}{2\pi n} \left(a_n c_n(t) + b_n s_n(t) \right) \end{aligned}}$$

Step 1 Regularise: cut-off Fourier series at $n \leq N$. (ultra-violet cut-off)

$$\int_0^1 \tilde{g}^N \, dg^N = \sum_{n=1}^{N} \frac{1}{2\pi n} \left(a_n^2 + b_n^2 \right)$$

$$\Rightarrow \mathbb{E} \int_0^1 \tilde{g}^N \, dg^N = \sum_{n=1}^{N} \frac{1}{\pi n} = \frac{\log N}{\pi} + O(1) = C_N$$

diverges logarithmically

Step 2 Renormalised integrals do converge

$$\boxed{\int_0^1 \tilde{g}^N \, dg^N - C_N \xrightarrow[\text{in } L^2]{} \text{"} \int \tilde{g} \, dg - \infty \text{"}}$$

"Wick power"

Example:

Ito calculus has many applications (pricing of options, large scale dynamics of particle models, ...)

Here MCMC algorithms:

Question: Given a (complicated) probability measure μ; how to compute $\int f \, d\mu$

nice test function.

Answer: Find a suitable stochastic process (Markov chain) X_s and compute.

$$\boxed{\frac{1}{T}\int_0^T f(X_s)\,ds}$$

Hope (often true - ergodic Thm)

$$\boxed{\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X_s)\,ds = \int f \, d\mu}$$

time average

Space average

Langevin dynamics:

normalisation constant

Lebesgue measure on $\mathbb{R}^d$

If $$\boxed{\mu(dx) = \frac{1}{Z} e^{-V(x)} dx}$$

Candidate dynamics: Langevin

$$\boxed{dX_t = -\nabla V(X_t)\, dt + \sqrt{2}\, dB_t}$$

- X_t is reversible under μ i.e.

process in equilibrium

$$\boxed{\mathbb{P}(X_t \in dx \;\&\; X_{t+s} \in dy) = \mathbb{P}(X_t \in dy \;\&\; X_{t+s} \in dx)}$$

- Advantage: no need to know Z.
- Default choice in many situations.
- Can be used in ∞ dimensions (e.g. Bayesian inference...)

Quantum Fields

- Quantum field theory (QFT) was developed to understand nature at smallest scales. The standard model of particle physics is a QFT.

- Highly successful theory, tested to extreme precision (e.g. anamalous magnetic moment of electron)

- Mathematically incomplete - to date no rigorous construction of a QFT in the physical 4 space-time dimensions.

cf. Yang-Mills Clay math problem.

Euclidean Quantum Field Theorys

One approach to construct a QFT leads through

- Construction of a measure on functions/distributions ϕ of the form

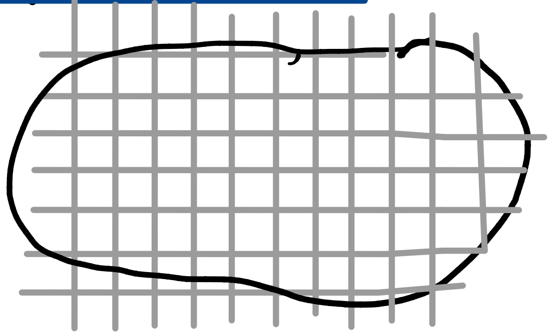
$$\text{“} \quad \mu(d\phi) \sim \exp\big(-S(\phi)\big) \prod_{x \in \mathbb{R}^d} d\phi_x \quad \text{”}$$

action

- **Example:**
  - $S(\phi) = \int m^2 \phi + |\nabla \phi|^2 \, dx$ “Free field”
  - $S(\phi) = \int m^2 \phi + |\nabla \phi|^2 + \phi^4 \, dx$ “ϕ^4 theory”

“easy theories”

· Interpretation Discrete, finite lattice



$\Lambda_\varepsilon = \Lambda \cap \varepsilon \mathbb{Z}^d$

$\phi : \Lambda_\varepsilon \to \mathbb{R}/\mathbb{C}$

finite dim. Lebesgue

$$\sim \exp\big(-S_\varepsilon(\phi)\big) \prod_{x \in \Lambda_\varepsilon} d\phi_x$$

Take limits: $\varepsilon \downarrow 0$ ultraviolet
$\Lambda \uparrow \mathbb{R}^d$ infrared

$\boxed{\text{Free field}}$ $\quad S(\phi) = \int m^2\phi + |\nabla\phi|^2 \, dx \qquad \mu(d\phi) \sim \exp\left(-S(\phi)\right) \prod_{x\in\mathbb{R}^d} d\phi_x$

Can be constructed / analysed easily:

E.g. on torus $\mathbb{T}^d$:

(almost) iid (complex) Gaussian.

$$\boxed{\phi(x) = \sum_{k\in\mathbb{Z}^d} \frac{1}{\sqrt{4\pi^2|k|^2+m^2}} e^{2\pi i k.x} X_k}$$

Always well-defined but very irregular:

$2s \quad < -d+2$

$$\mathbb{E}\|\phi\|^2_{H^s} = \sum_{k\in\mathbb{Z}^d} \frac{1}{4\pi^2|k|^2+m^2} (1+|k|^2)^s \underbrace{\mathbb{E}|X_k|^2}_{\sim 1} < \infty \iff \boxed{s < \frac{2-d}{2}}$$

Dimension d	Regularity s
1	$< 1/2$
2	< 0
3	$< -1/2$
4	< -1

rigorous interpretation - no problem.
but very irregular

the higher d, the worse!

$\boxed{\phi^4 \text{ theory}}$

- $S(\phi) = \int m^2\phi + |\nabla\phi|^2 + \phi^4 \, dx \qquad \mu(d\phi) \sim \exp(-S(\phi)) \prod_{x\in\mathbb{R}^d} d\phi_x$

Construction/analysis of measure:

$d=1$ immediate

$d=2$ Nelson 60s ← renormalisation needed

$d=3$ Glimm-Jaffe, ... 70s ← more renormalisation needed

$d\geq 4$ triviality results: $d\geq 5$ Aizenman '82, Fröhlich '82

$d=4$ Aizenman & Duminil-Copin 2021.

Watanabe '82 JSP "Euclidean ϕ^4_3 field theory has [...] become a mature branch of mathematical physics"

Stochastic quantisation

$\begin{cases} \text{Parisi-Wu (motivation in gauge fixing) '82} \\ \text{Creutz - Freedman (MCMC method) '82} \end{cases}$ ← 1641 Zitate

- Look at "Langevin dynamics" for "Euclidean Field theories".

Leads to stochastic (partial) differential equation

$$\boxed{\partial_t \phi = \underbrace{(\Delta\phi - \phi^4)}_{= -\nabla S(\phi)} + \xi}$$

white noise in d+1 space time dimensions

Attempts to develop a theory in d=2 in 80s & 90s. Resolution in d=2 by da Prato-Debussche '02.

The theory of regularity structures

for

Thm: (Hairer 2014) Short time well posedness for SPDEs including stochastic quantisation of Φ^4_3.

Regularised equation

$$\boxed{(\partial_t - \Delta)\phi_\varepsilon = -\left(\phi_\varepsilon^3 - \big(\overset{\sim \frac{1}{\varepsilon}}{3C_\varepsilon} + \overset{\sim \log \varepsilon^{-1}}{9\bar{C}_\varepsilon}\big)\phi_\varepsilon\right) + \xi_\varepsilon}$$

Up to a random time $\tau > 0$ ϕ_ε converges to non-trivial limit.

Thm: Mourrat-W. '17 Solutions on $(0,\infty)\times\mathbb{T}^3$ exist globally in time.

Estimates imply existence of invariant measures.

On $\mathbb{R}^3$: Gubinelli-Hofmanová 2019, Mourrat-Weber 2020.

Regularity structures in a nutshell

- For simplicity $\boxed{(\partial_t-\Delta)u = g(u)\xi \qquad x\in\mathbb{T}^2}$ (gPAM)

$$\boxed{\xi = \sum_{k\in\mathbb{Z}^2} e^{2\pi i x} X_k}$$

essentially indep. Gaussians.

Spacial white noise.

- Power counting: $\xi \in C^{a-2}$ $\quad a<1.$ $\rightsquigarrow$ parabolic regularity $\quad u$ should be in C^a

$$\boxed{a+a-2<0} \rightsquigarrow g(u)\xi \text{ not defined.}$$

- Freeze coefficients around base point $z_0=(t_0,x_0)$

$$(\partial_t-\Delta)u = g(u(z_0))\xi + \underbrace{(g(u)-g(u(z_0)))}_{\text{small near } z_0}\xi$$

Regularity structures 2

- Near z_0, u should "look like" sol'n of

$$(\partial_t-\Delta)u = g(u(z_0))\xi$$

$$(\partial_t-\Delta)u = g(u(z_0))\xi \quad + \underbrace{(g(u)-g(u(z_0)))}_{\text{small near } z_0}\xi$$

- Modelledness

$$\left| u(z)-u(z_0)-g(u(z))\big(Z(z)-Z(z_0)\big)+\dots \right| \lesssim d(z,z_0)^{2\alpha}$$

where $(\partial_t-\Delta)Z=\xi$ (Stochastic heat eqn.)

- to define $u\xi$ (or $g(u)\xi$) it suffices to construct

$$Z\xi$$

- A "renormalised" product "$Z\xi-\infty$" can be constructed just like renormalised stoch integral

$$(\partial_t - \Delta) u = g(u)\xi \qquad (t,x) \in (0,\infty) \times \mathbb{T}^d \qquad (gPAM)$$

"Noise" ξ: Space-time distribution of regularity $-1-\kappa$, $\kappa < 0{,}132\ldots$

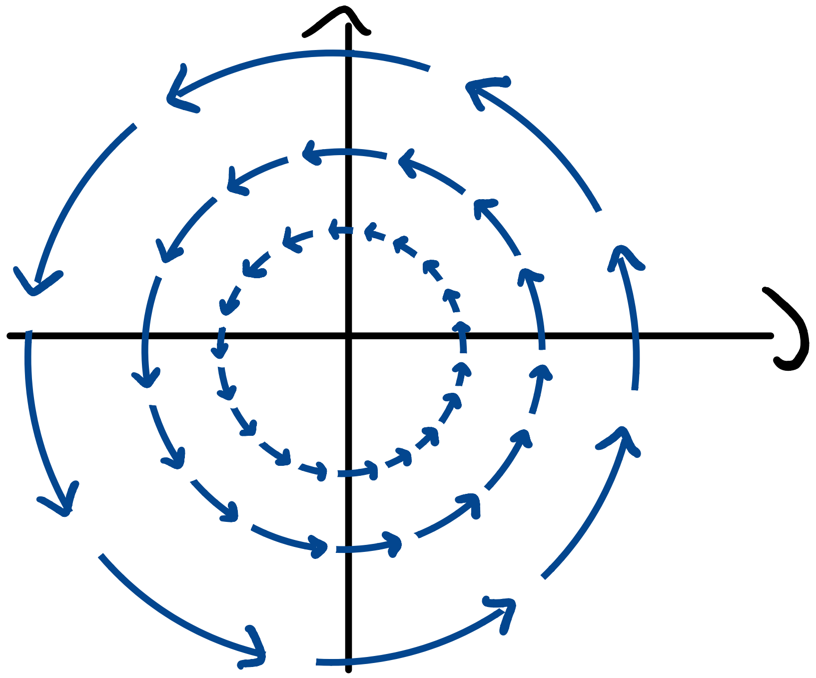
Example: ξ = space-white noise over $\mathbb{T}^2$

Coefficients g: $g : \mathbb{R} \to \mathbb{R}$ regular; bdd. $\|g\|; \|g'\|; \|g''\| < \infty$

Thm (Chandra, Feltes, W. 2024)

Estimate for $\|u(t)\|_{L^\infty}$ for $t \gtrsim 1$. $\Rightarrow$ u cannot explode in finite time

Global existence can fail in low regularity



$\in \mathbb{R}^2$

$\begin{pmatrix}x_1\\x_2\end{pmatrix}^\perp = \begin{pmatrix}-x_2\\x_1\end{pmatrix}$ rotation

$$\partial_t x = |x|^m x^\perp + \partial_t W \qquad (*)$$

- local existence easy for W continuous.
- global existence easy if W Lipschitz.

Thm. (Cox, Hutzenthaler, Jentzen '13): For $m \geq 3$ $\exists\, \alpha = \alpha(m) > 0$ and $W \in C^\alpha$ s.th. x blows up in finite time.

- Idea: $\partial_t W = |x|^\varepsilon x$ $\qquad \int_s^t |x|^\varepsilon x$ has cancelations due to fast rotation of x
- See also: Scheutzow, Engel, Chemnitz 2024+.