The Beautiful Mathematics Involved in the Study of Dispersive Equations

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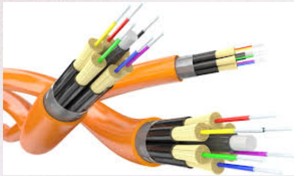
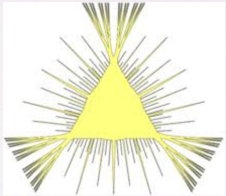
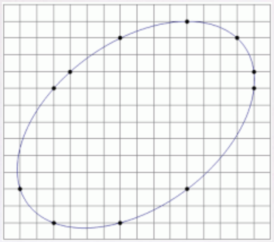
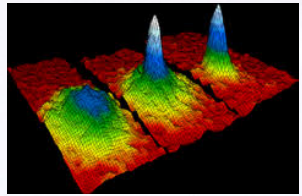
What do these pictures have in common?







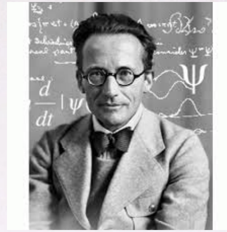


Tools come from many areas

The extraordinary recent progress in dispersive equations has involved:

* Harmonic and Fourier Analysis
* Analytic Number theory
* Math Physics
* Dynamical Systems
* Symplectic Geometry
* Probability

A case study: the nonlinear Schrödinger equation



$$(\mathrm{NLS}) \quad \begin{cases} i\partial_t u + \Delta u = \lambda |u|^2 u & \lambda = \pm 1 \\ u(0,x) = u_0(x) & x \in \mathbb{T}^d \end{cases}$$

this is the periodic NLS initial value problem.

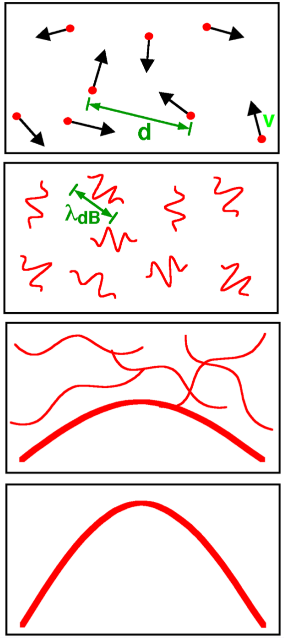
$$\text{Mass} = \int_{\mathbb{T}^d} |u(t,x)|^2 \, dx$$

$$\text{Hamiltonian} = \int_{\mathbb{T}^d} \frac{1}{2} |\nabla u(t,x)|^2 + \frac{\lambda}{4} \int_{\mathbb{T}^d} |u(t,x)|^4 \, dx$$

these integrals are conserved!

Where does the NLS come from?

What is Bose-Einstein condensation (BEC)?

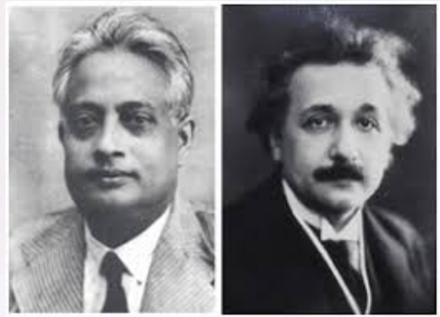


High Temperature T:
thermal velocity v
density d^{-3}
"Billiard balls"

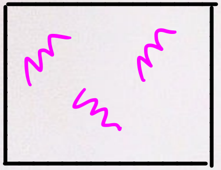
Low Temperature T:
De Broglie wavelength
$\lambda_{dB} = h/mv \propto T^{-1/2}$
"Wave packets"

$T = T_{crit}$:
Bose-Einstein Condensation
$\lambda_{dB} \approx d$
"Matter wave overlap"

T=0:
Pure Bose condensate
"Giant matter wave"

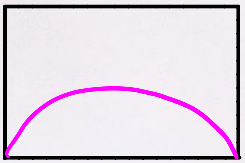


Mathematically:



"wave packets" $\Longrightarrow$ The BBGKY hierarchy

$\Big\Downarrow$ take limit as $N \to \infty$



"giant matter wave" $\Longrightarrow$ The Gross-Pitaevskii hierarchy

More Mathematics

If $\underline{x}_k = (x_1, \dots x_k)$ $\quad x_i \in \mathbb{R}^d, \mathbb{T}^d$

$$\boxed{\gamma_0^{(k)}(\underline{x}_k, \underline{x}_k') = \prod_{j=1}^{k} u_0(x_j)\,\overline{u_0(x_j')}}$$

(initial data of G-P)

then

$$\boxed{\gamma^{(k)}(t, \underline{x}_k, \underline{x}_k') = \prod_{j=1}^{k} u(t, x_j)\,\overline{u(t, x_j')}}$$

(solution to G-P)

where

$$(\text{NLS}) \begin{cases} i\partial_t u + \Delta u = |u|^2 u \\ u(0,x) = u_0(x) \end{cases}$$

Spohn

Erdös - Schlein - Yau

Kirkpatrick - Schlein - S.

T. Chen - Pavlovic

X. Chen - Holmer

Example of an integrable system

$$i\partial_t u + \partial_x^2 u = \pm |u|^2 u$$

in $\mathbb{R}$, $\mathbb{T}$

is an $\boxed{\text{integrable system}}$

Lax Pairs, Inverse Scattering, infinitely many conservation laws:

$$I_s(u) = \int \frac{1}{2} |D^s u|^2 dx + \text{l.o.t.}$$

$s \in \mathbb{N}$

Gross - Pitaevskii Hierarchy in $\mathbb{R}$, $\mathbb{T}$ also admits infinitely many $\boxed{\text{conserved quantities}}$

$\boxed{\text{Mendelson - Nahmod - Pavlović - S}}$

Note: We expect even more structure.

Hamiltonian Structure and Poisson Commuting Energies

Many Body system — BBGKY

$\Downarrow$ $N \to \infty$

GP hierarchy

$\Updownarrow$

Schrödinger Equation

What is here?

? $\Downarrow$

What is here?

$\Downarrow$?

Hamiltonian structure

Canonical Poisson structure

If in $\mathbb{R}$ and cubic $\Rightarrow$ geometric integrable structure (Palais)

Can the GP be realized as a Hamiltonian equation of motion with H_{GP} on some Poisson manifold?

Can the Poisson structure and Hamiltonian H_{GP} be derived in a suitable sense from an analogous structure at the N-particle level?

In 1D does the cubic GP possess an integrable structure in the sense that $\exists \{H_n\}_{n\in\mathbb{N}}$ that Poisson commutes and that contains H_{GP}?

If $\{H_n\}_{n\in\mathbb{N}}$ exists does each of the H_n generate a Hamiltonian equation of motion related to the known n^{th} Schrödinger equation?

the answers are Yes for all the questions above!

(D. Mendelson, A. Nahmod, N. Pavlovic, M. Rosenzweig, G.S.)

Well-Posedness

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^2 u \\ u(0,x) = u_0(x) \end{cases} \iff u(t,x) = S(t)u_0(x) \pm \int_0^t S(t-t') |u|^2 u(t') dt'$$

$S(t)u_0(x)$ = solution to linear Schrödinger:

$$\begin{cases} i\partial_t v + \Delta v = 0 \\ v(0,x) = u_0(x) \end{cases}$$

Solution to Cauchy problem $\iff$ Fixed point of integral equation.

Periodic Strichartz Estimates

We need a good Banach space for a fixed point argument. The Strichartz Estimates on $S(t)u_0(x)$ help:

$$\|S(t)u_0\|_{L^q_{[0,1]} L^q_{\mathbb{T}^d}} \le C \|u_0\|_{H^s(\mathbb{T}^d)}$$

$\boxed{\text{Case}: d=2,\ q=4}$

$$S(t)u_0(x) = \sum_{n \in \mathbb{Z}^2} \hat{u}_0(n)\, e^{it(\alpha_1 n_1^2 + \alpha_2 n_2^2)}\, e^{in\cdot x}$$

$\alpha_1, \alpha_2 > 0$

- $\alpha_1/\alpha_2 \in \mathbb{Q} \iff \mathbb{T}^2$ $\boxed{\text{rational}}$ torus
- $\alpha_1/\alpha_2 \notin \mathbb{Q} \iff \mathbb{T}^2$ $\boxed{\text{irrational}}$ torus.

Strichartz Estimates on rational tori

If $\mathbb{T}^2$ is rational torus then

Bourgain 90's

$$\boxed{\|S(t)u_0\|_{L^4_{\mathbb{T}\times\mathbb{T}^2}} \le C\,\|u_0\|_{H^\varepsilon(\mathbb{T}^2)}} \qquad \varepsilon > 0$$

Ingredients:

a) $\mathbb{T}^2$ rational $\Rightarrow$ $S(t)u_0(x)$ is also periodic in time.

For example take $\alpha_1, \alpha_2 \in \mathbb{N}$

$$S(t)u_0(x) = \sum_{n\in\mathbb{Z}^2} \hat{u}_0(n)\, e^{it(\alpha_1 n_1^2 + \alpha_2 n_2^2) + i n\cdot x}$$

e $\rightarrow$ time periodicity

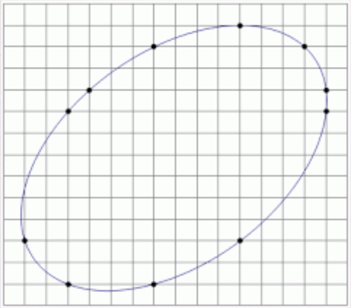
b) If π^2 is rational one can count the set

$$\boxed{\{n \in \mathbb{Z}^2 / \alpha_1 n_1^2 + \alpha_2 n_2^2 = R^2\} = \Sigma}$$

$\Downarrow \quad \swarrow$

$\alpha_1, \alpha_2, R \in \mathbb{N}$

$\nearrow \quad \Uparrow$



In fact

$$|\Sigma| \approx \exp c \frac{\log R}{\log \log R} \ll R^{\varepsilon}$$

(Gauss lemma)

$$\boxed{\text{Analytic Number theory} \Longrightarrow \text{Harmonic Analysis}}$$

Strichartz Estimates on any Torus

$$\|S(t)u_0\|_{L^4_{[0,1]}L^4_{\mathbb{T}^2}} \le C\|u_0\|_{H^\varepsilon} \quad \varepsilon > 0$$

Bourgain - Demeter '14

Surprisingly ANT was not part of the proof. It is in fact consequence of the

ℓ^2 Decoupling Theorem

this theorem had been a major conjecture in HA. It is related to the Fourier Restriction theorem and the Kakeya problem.

Improvements and consequences

* A bilinear Strichartz estimate was proved by C. Fan - S-H. Weng - B. Wilson
* Longer time Strichartz estimates were proved by Y. Deng - P. Germain - L. Guth.
* Sharp Decoupling for curves $\Longrightarrow$ Vinogradov Mean Value theorem
  J. Bourgain - C. Demeter - L. Guth

Harmonic Analysis $\Longrightarrow$ Analytic Number Theory

Global well-posedness and properties

Now using Strichartz estimates and a fixed point argument one can claim that the Cauchy problem:

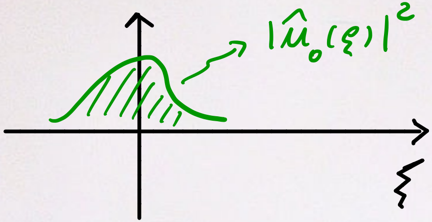
$$\begin{cases} i\partial_t u + \Delta u = \lambda |u|^2 u \\ u|_{t=0} = u_0(x) \quad x \in \mathbb{T}^2 \end{cases} \qquad \lambda = \pm 1$$

is locally well-posed in $H^s(\mathbb{T}^2)$, $s > 0$. If $\lambda = 1$ (defocusing) then energy conservation $\Rightarrow$ global well-posedness for $s \geq 1$.

<u>Question</u>: Can we learn more about the behaviour of the solution $u(t,x)$ as $t \to \infty$?

Transfer of energy

$t = 0$ $t > 0$



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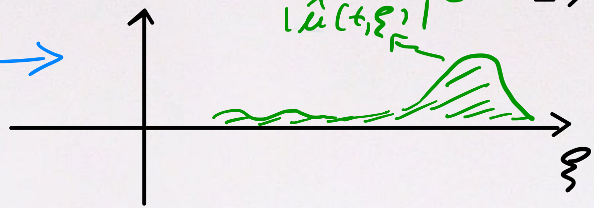
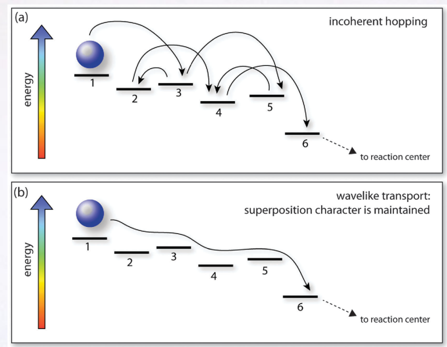


Question: 1) Does the support of $|\hat{u}(t,\xi)|$ move to higher frequencies?

(Weak turbulence, forward cascade)

2) If such a "migration" happens, is it done in a incoherent hopping way or more like a wavelike transport?



What we can say mathematically

One possible way to investigate 1) is looking at

$$\boxed{\int |\hat{u}(t,\xi)|^2 (1+|\xi|)^{2s}\, d\xi =: \|u(t)\|^2_{H^s}}$$

and checking

$$\boxed{\lim_{t\to\infty} \|u(t)\|_{H^s}}$$

this is the problem of Growth of Sobolev Norms.

By iteration of local estimates $\Longrightarrow$ $\|u(t)\|_{H^s} \le C_1 e^{c_2|t|}$.

Can one do better?

Growth of Sobolev Norms

Fact 1: Complete integrability may prevent the growth of Sobolev norms (1D cubic NLS, KdV)

Fact 2: Scattering prevents the growth of Sobolev norms:

(Defocusing cubic NLS in $\mathbb{R}^2$: If $u(t,x)$ is solution in $H^s(\mathbb{R}^2)$ then $\exists\, u^+ \in H^s(\mathbb{R}^2)$ s.t.

$$s \geq 0 \qquad \boxed{\| S(t)u^+ - u \|_{H^s} \xrightarrow[t \to +\infty]{} 0}$$

(Dodson '16)

As a consequence for $t \gg 1$

S(t) is unitary!

$$\| u(t) \|_{H^s} \leq \| S(t) u^+ - u \|_{H^s} + \| S(t) u^+ \|_{H^s} \leq \varepsilon + \| u^+ \|_{H^\varepsilon}.$$

Some bounds from above

Assume $u(t,x)$ is the global smooth solution to

$$\begin{cases} i\partial_t u + \Delta u = |u|^2 u \\ u|_{t=0} = u_0 \end{cases}$$

Fact 1: If $x \in \mathbb{T}^2$ then

$$\Longrightarrow \boxed{\|u(t)\|_{H^s} \leq C\,|t|^{2(s-1)+\varepsilon} \qquad |t| \geq 1}$$

[Bourgain, Sohinger)

Fact 2: Consider the NLS with nonlinearity $|u|^{p-1}u$, $3<p<5$ in generic Tori $\mathbb{T}^3$. Then one has

$$\boxed{\|u(t)\|_{H^2(\mathbb{T}^3)} \leq C\,(1+|t|)^{\frac{2}{5-p}+\theta(p)}}$$

For rational $\mathbb{T}^3$ it would not be here!

$$\theta(p) = \frac{\min(p-3,\, 5-p)}{182}$$

(Y. Deng - P. Germain)

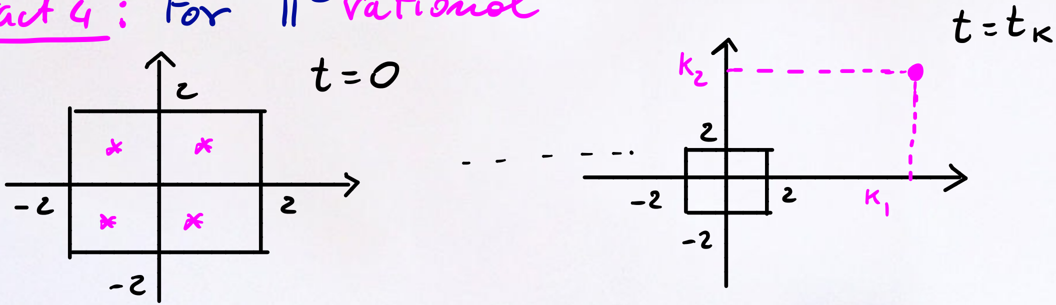
Are there solutions that grow?

Fact 3: Fix $s>1$, $0<\delta \ll 1$, $K \gg 1$, then for the cubic, defocusing NLS in $\mathbb{T}^2$ rational, $\exists$ an initial data $u_0 \in H^s$ and a time $T \gg 1$ s.t.

$$\|u_0\|_{H^s} < \delta \quad \text{and} \quad \|u(T)\|_{H^s} > K$$

(Colliander-Keel-S-Takaoka-Tao)

Fact 4: For $\mathbb{T}^2$ rational



arbitrarily large modes exists.

(Carles-Faou)

Some ideas of the proof of Fact 3

This is a constructive proof. Assume $\mathbb{T}^2$ is square.
Look for a solution

$$u(t,x) = \sum_{n \in \mathbb{Z}^2} a_n(t)\, e^{i(t|n|^2 + x\cdot n)} \iff$$

$$\boxed{-i\partial_t a_n = -|a_n|^2 a_n + \sum_{n_1,n_2,n_3 \in \Gamma(n)} a_{n_1}\bar{a}_{n_2} a_{n_3} e^{i\omega_4 t}} \quad n \in \mathbb{Z}^2$$

where

$$\omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2$$

$$\Gamma(n) = \{(n_1,n_2,n_3) \,/\, n_1 - n_2 + n_3 = n\}$$

this is a HUGE system!

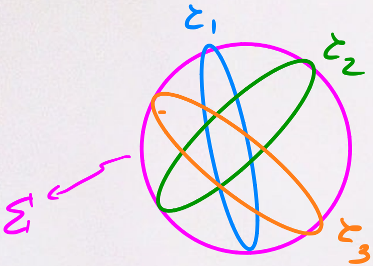
Toy Model

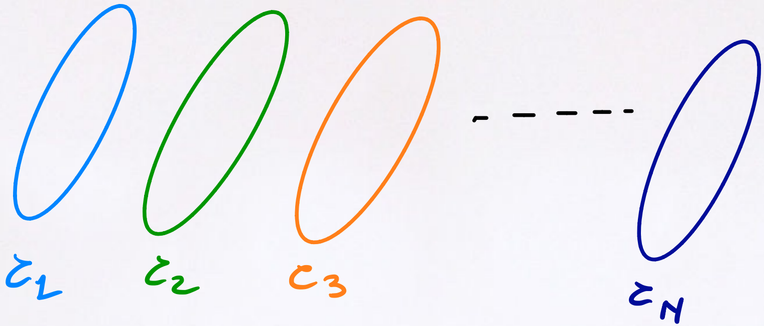
$$\begin{cases} -i\,\dot{b}_j = -|b_j|^2 b_j + 2\, b_{j-1}^2\, \bar{b}_j + 2\, b_{j+1}^2\, \bar{b}_j & j = 1, \dots, N \\ b_1(t) = b_N(t) = 0 & \rightsquigarrow \text{boundary data} \\ b_j(0) = \tilde{b}_j & \rightsquigarrow \text{initial data} \end{cases}$$

Remark: Although this is not the original system, one can prove that its solution approximate well the one of the original NLS system.

The dynamics

Conservation of mass $\implies$ $\Sigma' = \{ x \in \mathbb{C}^N / |x|^2 = 1 \}$ is where the dynamics happens

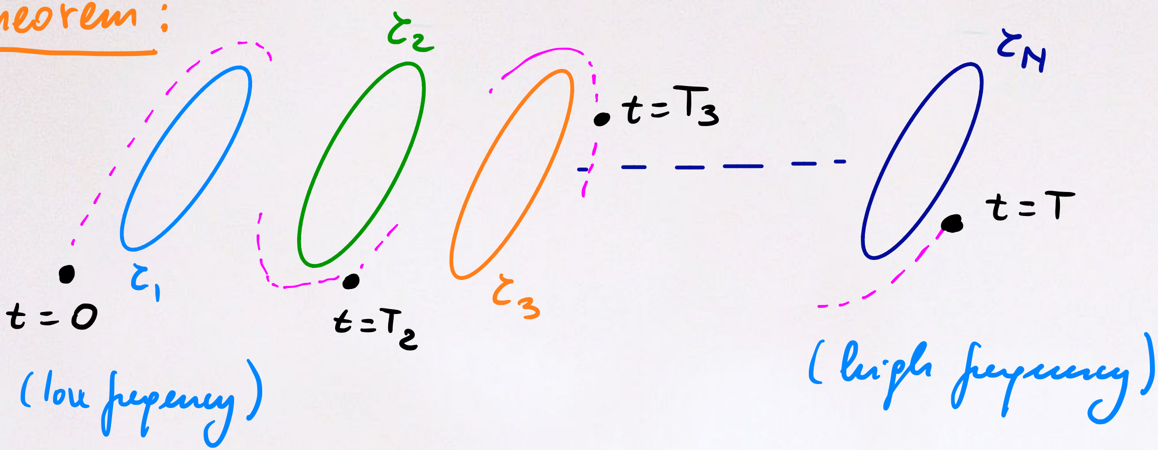




on Σ' there are τ_j, $j = 1, \ldots, N$, great circles that are invariant.

The heart of the matter

Theorem :



See also a more Dynamical System approach from Guardia - Kaloshin, Haus - Procesi

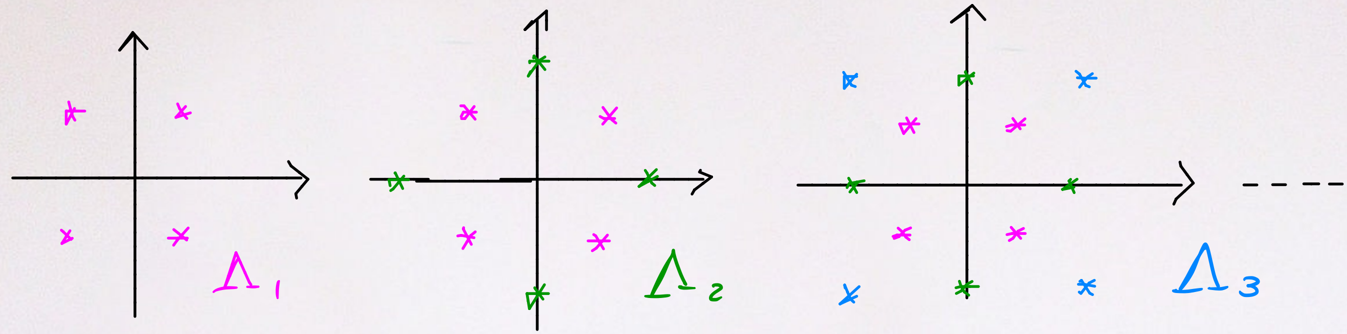
Some Remarks

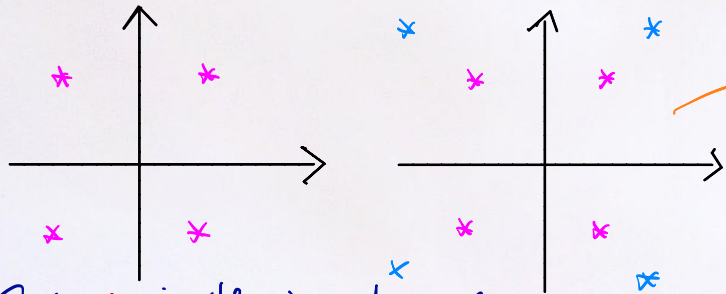
* We do not know what happens after time T.
* In the work of Carles-Faou the procedure is different but the same set Λ of frequencies is used.

Question: What happens when π^2 is irrational?

Answer: In collaboration with B. Wilson we proved that the dynamics exploited by C-K-S-T-T and C-F cannot happen.

Set Δ in the rational case:





Diamonds are not allowed in the resonant set of irrational tori!

Set Δ in the irrational case

the NLS as ∞ dim Hamiltonian system

$$\begin{cases} i\partial_t u + \Delta u = |u|^2 u \\ u|_{t=0} = u_0 \\ x \in \mathbb{T}^d \end{cases}
\quad \xleftrightarrow{\text{Fourier Transform}} \quad
\begin{cases} \dot{a}_n = \dfrac{\partial H}{\partial b_n} \\ \dot{b}_n = -\dfrac{\partial H}{\partial a_n} \\ n \in \mathbb{Z}^d \end{cases}$$

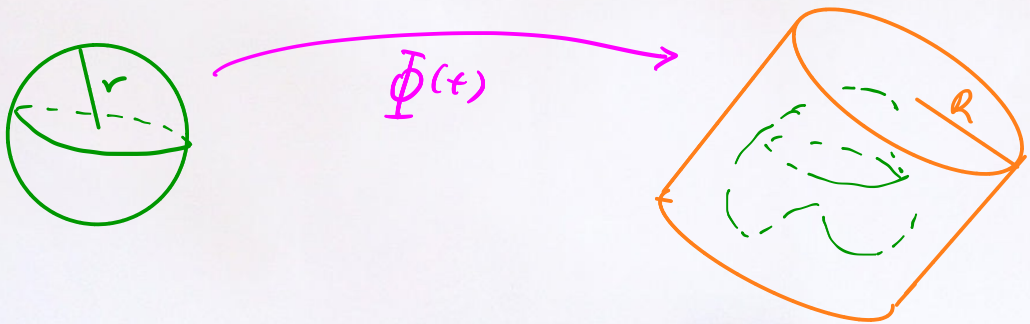
$$\boxed{\hat{u}(t,n) = a_n(t) + i\, b_n(t)}$$

Question: Are theorems for a finite dim Hamiltonian system true for an infinite dim one?

Answer: It depends on the theorem.

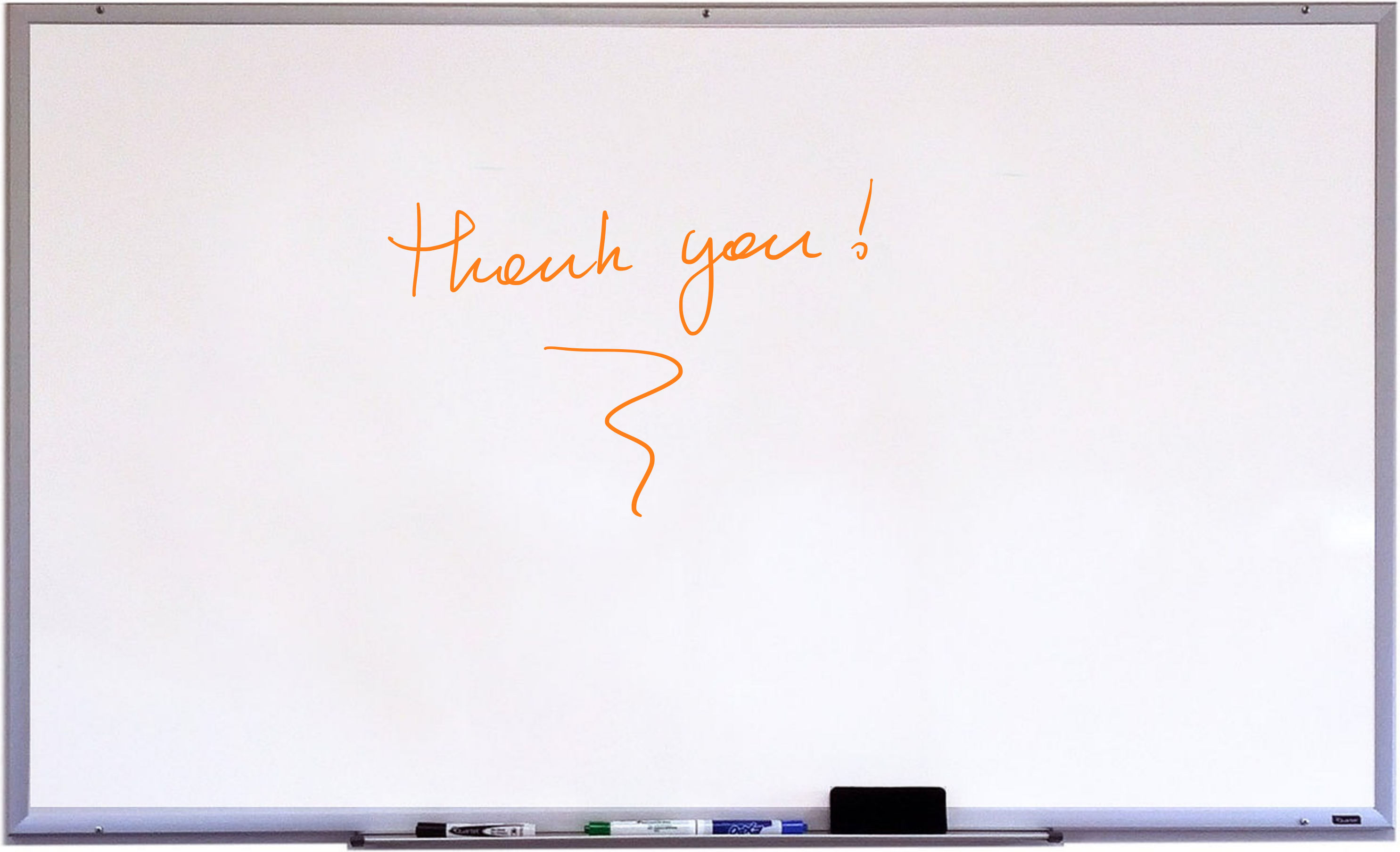
The nonsqueezing theorem

Theorem (Gromov) Assume $\Phi(t)$ is an Hamiltonian flow in $\mathbb{R}^{2d}$ which is also a symplectomorphism. Let B_r bee ball of radius r in $\mathbb{R}^{2d}$ and C_R be a cylinder of radius R in $\mathbb{R}^{2d}$. Then if $\Phi(t)(B_r) \subseteq C_R \implies r \leq R$



$\boxed{\text{Is a non-squeezing true for dim} = \infty \text{ flows?}}$

* True if $\Phi(t)$ is a compact perturbation of a linear flow (Kuksin).
* True for the cubic defocusing NLS in $\mathbb{T}$ (Bourgain) (Here $L^2(\mathbb{T})$ is the symplectic space)
* True for KdV in $H^{-\frac{1}{2}}(\mathbb{T})$ (Colliander-Keel-S-Takaoka-Tao)
* Partial results for Klein-Gordon almost sure flow in $H^{\frac{1}{2}} \times H^{-\frac{1}{2}}(\mathbb{T}^3)$ (Mendelson)
* True for cubic, defocusing NLS in $L^2(\mathbb{R}^2)$ (Killip-Visan-Zhang)


Thank you!