Classifying simple amenable C^*-algebras

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Mathematics Muenster: Mid Term Conference

Operator algebras

An example

- $\mathcal{H}$ a Hilbert space, a complete inner product space.
- $\mathcal{B}(\mathcal{H})$ the continuous linear operators on $\mathcal{H}$.
- Algebraic structure. *-algebra: $\langle T^*\xi, \eta\rangle = \langle \xi, T\eta\rangle$.
- Analytic structure. $\|T\| = \sup\{\|T\xi\| : \xi \in \mathcal{H},\ \|\xi\| \leqslant 1\}$.
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- *-subalgebras of $\mathcal{B}(\mathcal{H})$ closed in norm topology;

von Neumann algebras

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C*-algebras

- *-subalgebras of $\mathcal{B}(\mathcal{H})$ closed in norm topology;
- Commutative algebras, $C_0(X)$, locally compact X
- Topological nature

von Neumann algebras

- *-subalgebras of $\mathcal{B}(\mathcal{H})$ closed under pointwise limits
- Commutative algebras, $L^\infty(X)$, measure space X
- Measure theoretic nature

Structure and classification

Classification

- of classes of operator algebras upto isomorphism
- invariants computable in natural examples

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- Abstractly identify classifiable classes
- Reap structural benefits from classification

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C^*-algebras

- ‘Elliott programme’ – large scale project seeks analogous results
- Work of many researchers over decades

Examples from group actions

- Group action $\beta : G \curvearrowright X$.
- Induces action on functions $\alpha : G \curvearrowright C(X)$

$$\alpha_g(f)(x) = f(\beta_g^{-1}(x))$$

Eg: irrational rotation

- $\mathbb{Z} \curvearrowright \mathbb{T}$ by rotation by an irrational multiple θ of 2π.
- Space of orbits $\mathbb{T}/\mathbb{Z}$ badly behaved.

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In the spirit of the semi-direct product for groups

- Embed $C(X) \subseteq C(X) \rtimes_\alpha G$ in a larger algebra, so the action $\alpha : G \curvearrowright C(X)$ becomes inner in this larger algebra.
- $C(X) \rtimes_\alpha G$ a non-abelian C^*-algebra generated by $C(X) \subset \mathcal{B}(\mathcal{H})$, and unitaries u_g on $\mathcal{H}$ implementing the action.

Irrational rotation algebra $A_\theta = C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$

- C^*-algebra generated by unitaries U, V with $UV = e^{2\pi i\theta} VU$.
- Think of as a non-commutative torus.

Examples from group actions

Special case: Group operator algebras from unitary representations

X is a singleton: $C(X) = L^\infty(X) = \mathbb{C}$

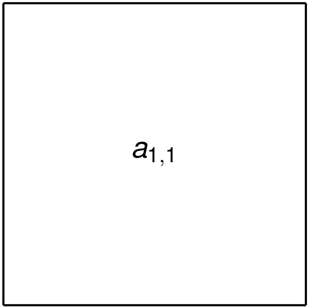
- $\mathbb{C} \rtimes G =: C_r^*(G)$ generated by a unitary representation of G on a Hilbert space (the left-regular representation).
- von Neumann version: $VN(G)$.

This generalises the Fourier transform for locally compact abelian groups

- G abelian: $C_r^*(G) = C_0(\widehat{G}),\ VN(G) = L^\infty(\widehat{G})$
- $C_r^*(\mathbb{Z}) = C(\mathbb{T}) \ncong C(\mathbb{T}^2) = C_r^*(\mathbb{Z}^2)$.
- $VN(\mathbb{Z}) = L^\infty(\mathbb{T}) \cong L^\infty(\mathbb{T}^2) = VN(\mathbb{Z}^2)$.

Examples: inductive limits

$\mathbb{C} =$



$\mathbb{C}$

Examples: inductive limits

$$M_2 \supset \mathbb{C} = \begin{pmatrix} a_{1,1} & 0 \\ 0 & a_{1,1} \end{pmatrix}$$

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EXAMPLES: INDUCTIVE LIMITS

$$M_4 \supset M_2 = \begin{array}{|c|c|c|c|}\hline a_{1,1} & 0 & a_{1,2} & 0 \\ \hline 0 & a_{1,1} & 0 & a_{1,2} \\ \hline a_{2,1} & 0 & a_{2,2} & 0 \\ \hline 0 & a_{2,1} & 0 & a_{2,2} \\ \hline \end{array}$$

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$$M_4 = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$\mathbb{C} \subset M_2 \subset M_4 \subset M_8 \subset \ldots$$

Examples: inductive limits

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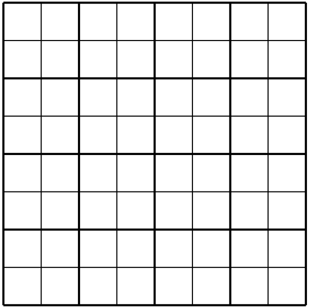

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This is all compatible with the normalised trace on these matrices:

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NOTICE

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$$\frac{1}{2}(a_{1,1} + a_{2,2}) = \frac{1}{4}(a_{1,1} + a_{1,1} + a_{2,2} + a_{2,2})$$

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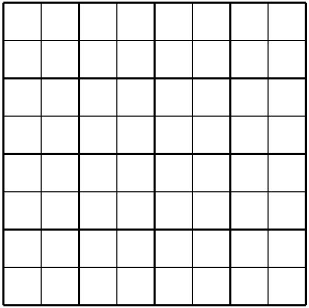
$M_8 =$ 

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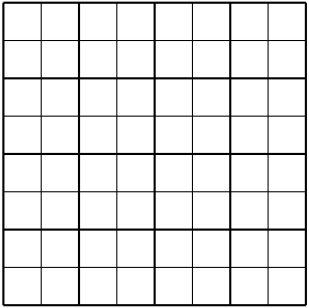
$M_8 =$  $\mathbb{C} \subset M_2 \subset M_4 \subset M_8 \subset \ldots$

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Represent on a Hilbert space and close to obtain:

- a C^*-algebra M_{2^∞} — the CAR algebra from mathematical physics.
- and a von Neumann algebra $\mathcal{R}$. The trace extends to these algebras.

Structure and Classification of vNAs

Theorem (Murray and von Neumann '45)

There exists a unique hyperfinite infinite dimensional simple von Neumann algebra with a trace acting on a separable Hilbert space.

- This is $\mathcal{R}$.

- Simple = no non-trivial von Neumann algebra ideals.
- infinite dimensional, simple, with a trace = 'II_1 factor'
- Hyperfinite: arises as an inductive limit of finite dimensional algebras as the previous example.

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Connes '77

Abstract characterisation of hyperfiniteness: amenability

- in terms of an operator algebraic version of amenability for groups.
- readily verifiable in examples
- $L^\infty(X) \rtimes G$ hyperfinite for G amenable (eg $G = \mathbb{Z}$).

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Structure (Connes) & classification (Murray-von Neumann)

There exists a unique (separably acting) amenable II_1 factor.

- Classification for traceless hyperfinite factors completed by Haagerup.
- Completely understand amenable vNas.

One of many ingredients in Connes work

- View our inductive limit construction of $\mathcal{R}$ as a representation of the infinite tensor product $\bigotimes_{\mathbb{N}} M_2$
- Of course $\bigotimes_{\mathbb{N}} M_2 \cong (\bigotimes_{\mathbb{N}} M_2) \otimes (\bigotimes_{\mathbb{N}} M_2)$
- This persists in the von Neumann tensor product: $\mathcal{R} \cong \mathcal{R} \otimes \mathcal{R}$.

Step in Connes proof:

A (separably acting) amenable II_1 factor $\mathcal{M}$ is McDuff if $\mathcal{M} \cong \mathcal{M} \otimes \mathcal{R}$

- $\mathcal{R}$ is acting as a tensorial unit on $\mathcal{M}$

Invariants for C^*-algebras

Can see the matrix size in the C^*-inductive limits

- $M_{2^\infty} \ncong M_{3^\infty}$
- $\because \tau(p) = \tau(q)$ when p and q are norm close projections can not approximate $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ by a projection in M_{3^n}.

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Making this less ad hoc: K-theory for C^*-algebras

- Non-commutative extension of Atiyah and Hirzebruch's K-theory for spaces
- For A unital, $K_0(A)$ constructed from equivalence classes of projections in matrices over A.
-
$$\begin{aligned} K_0(M_{n^\infty}) &= \{\tfrac{r}{n^k} : r \in \mathbb{Z},\ k = 0, 1, 2, \dots\} \\ &= \{\tau(p) - \tau(q) : p, q \text{ projections in matrices over } M_{n^\infty}\} \end{aligned}$$

  together with $[1_{M_{n^\infty}}]_0$ which corresponds to 1.

Traces: non commutative invariant measures

- M_{2^∞} has a unique trace, as each M_{2^n} does.

Traces on $C(X) \rtimes_\alpha G$

- Given by invariant measures on X (when action is essentially free)
- Collection of all traces is compact, convex.

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Irrational rotation by θ on $\mathbb{T}$

- Unique invariant measure — unique trace on $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$.
- $K_0(C(\mathbb{T}) \rtimes_\theta \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, $K_1(C(\mathbb{T}) \rtimes_\theta \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$

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- The pairing with the trace identifies

$$\tau(K_0(C(\mathbb{T}) \rtimes_\theta \mathbb{Z})) = \mathbb{Z} + (\theta/2\pi)\mathbb{Z} \subset \mathbb{R}.$$

- Irrational rotation algebras associated to θ_1 and θ_2 are isomorphic if and only if $\theta_2 = \pm\theta_1 \mod 2\pi\mathbb{Z}$.

Elliott programme

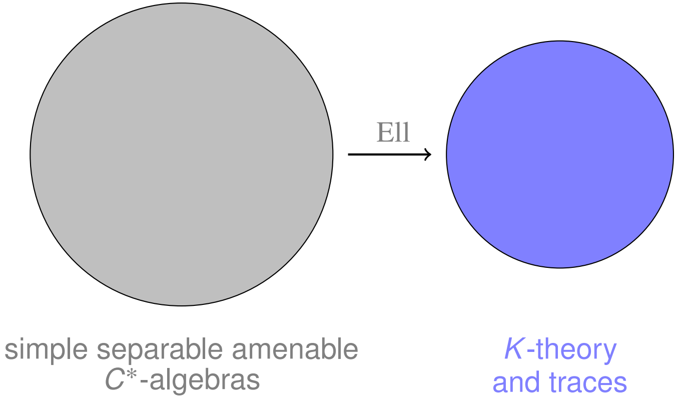
Classify simple separable amenable C^*-algebras by K-theory and traces

Amenability

- C^*-algebraic version of Connes' von Neumann algebraic amenability condition.
- readily testable in examples
- for G countable discrete, $C_r^*(G)$ amenable iff G is amenable.

ELLIOTT PROGRAMME

Classify simple separable amenable C^*-algebras by K-theory and traces



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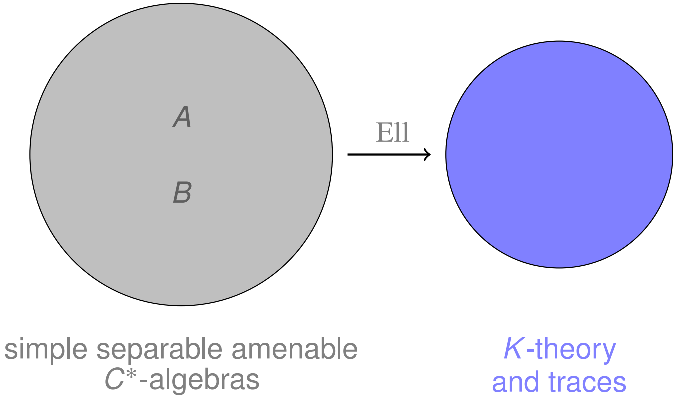

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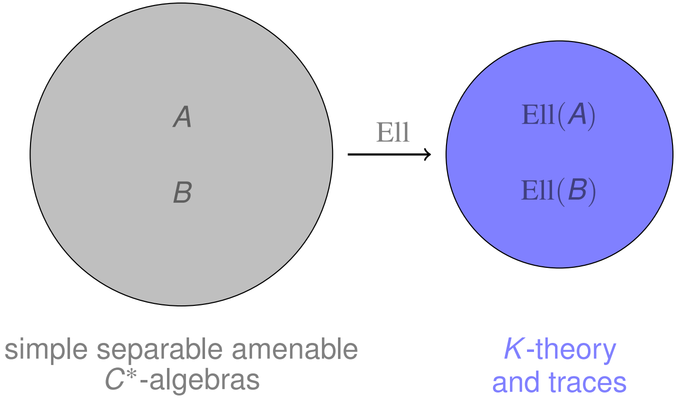

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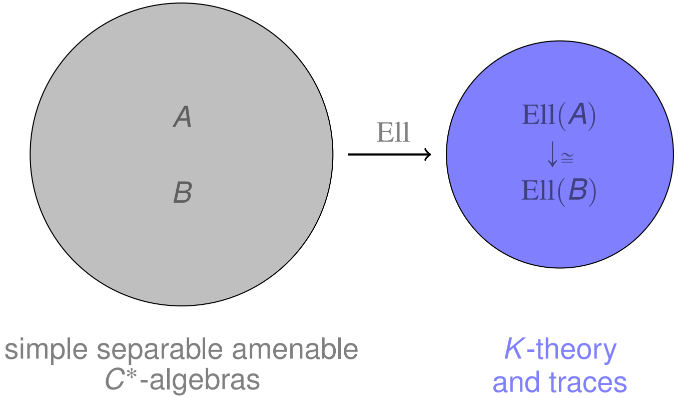

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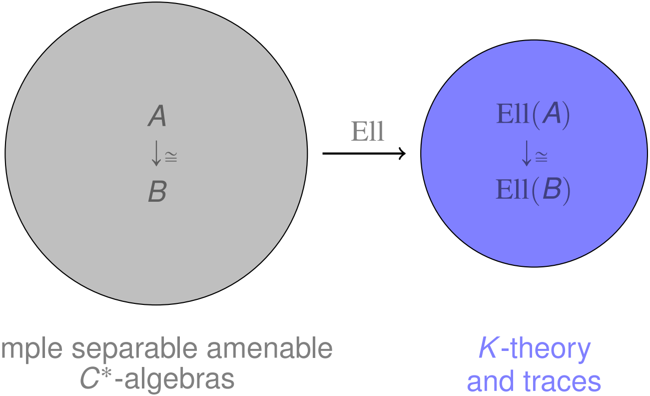

CLASSIFIABLE = ISOMORPHISMS LIFT

Every $\Phi : \mathrm{Ell}(A) \xrightarrow{\cong} \mathrm{Ell}(B)$ is $\mathrm{Ell}(\phi)$ for a (suitably) unique $\phi : A \xrightarrow{\cong} B$.

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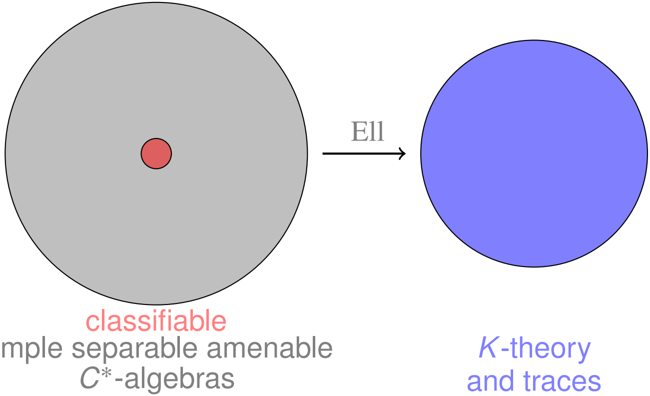

Aspects of the classifiable bubble:

- AF algebras (Elliott '76),

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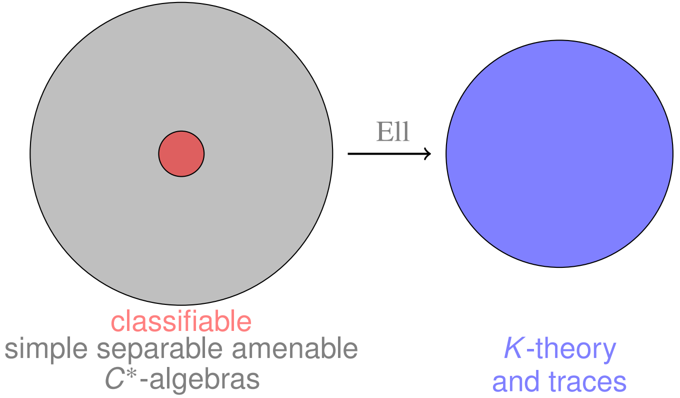

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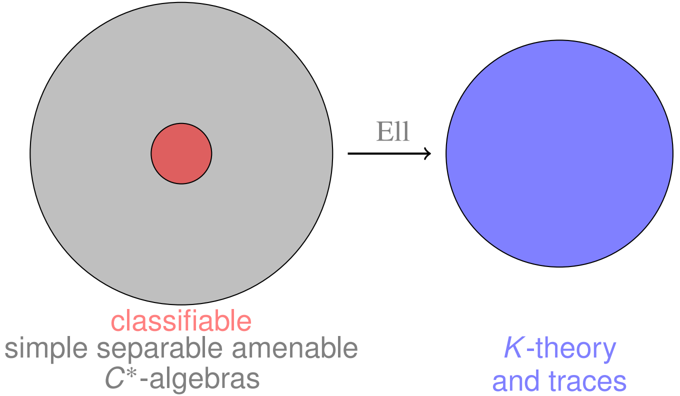

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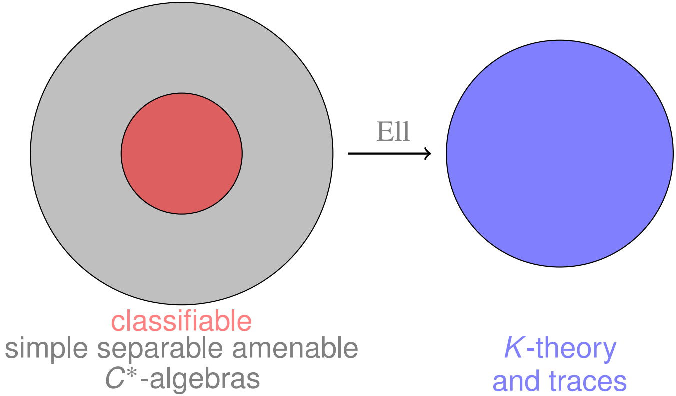

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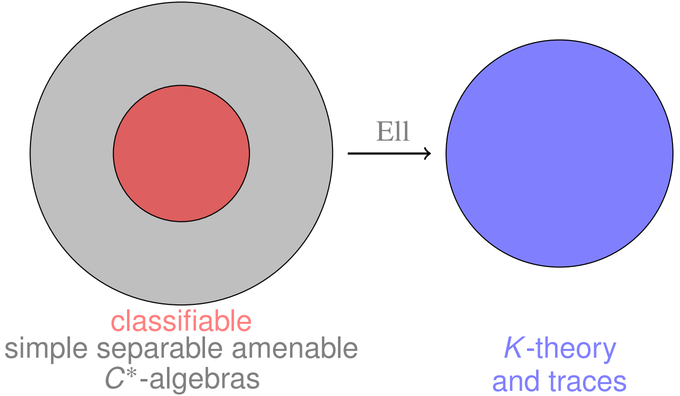

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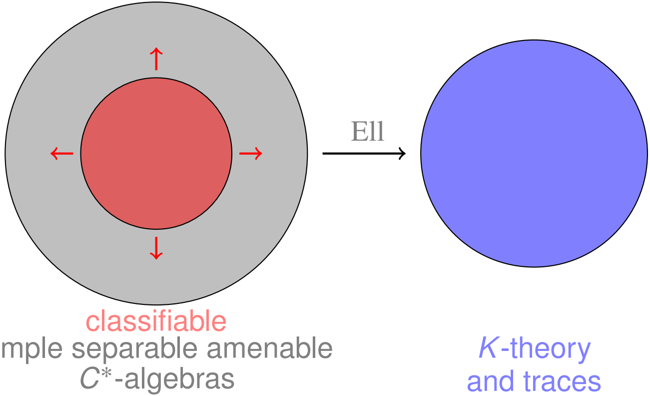

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Higher dimensional examples

Counter examples 2000s

Exist simple inductive limit A of C^*-algebras $M_{m_n}(C(X_n))$ such that $A \ncong A \otimes M_{2^\infty}$ but this can not be seen via K-theory and traces, or countably many other homotopy invariant functors into abelian groups

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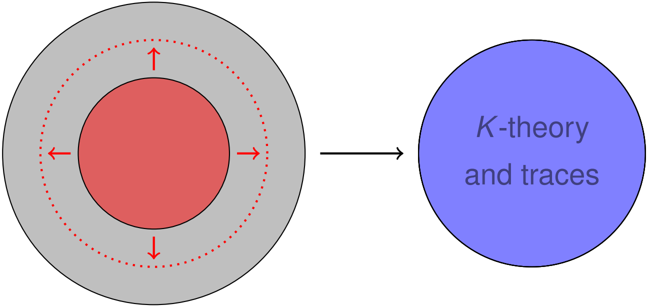

Question

Where is the dividing line between the classifiable and the exotic?

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Recall:

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WHAT IS THE RIGHT C^*-ANALOG OF $\mathcal{R}$?

RECALL:

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NEW EXAMPLE: JIANG-SU ALGEBRA $\mathcal{Z}$ FOUND LATE 90S

- Infinite dimensional simple separable unital amenable C^*-algebra with $\mathrm{Ell}(\mathcal{Z}) \cong \mathrm{Ell}(\mathbb{C})$.
- Construction somewhat intricate, but by now lots of different constructions all giving the same algebra.
- Can not have both $\mathbb{C}$ and $\mathcal{Z}$ within a class of algebras classified by K-theory and traces.

$\mathcal{Z}$ as a non-commutative tensor units

- A and $A \otimes \mathcal{Z}$ indistinguishable by K-theory and traces.

$\mathcal{Z}$-stability: $A \cong A \otimes \mathcal{Z}$

- $\mathcal{Z}$-stability a minimal non-trivial absorption hypothesis.
- There are efficient tools for describing $\mathcal{Z}$-stability (without reference to $\mathcal{Z}$) which in spirit go back to McDuff.

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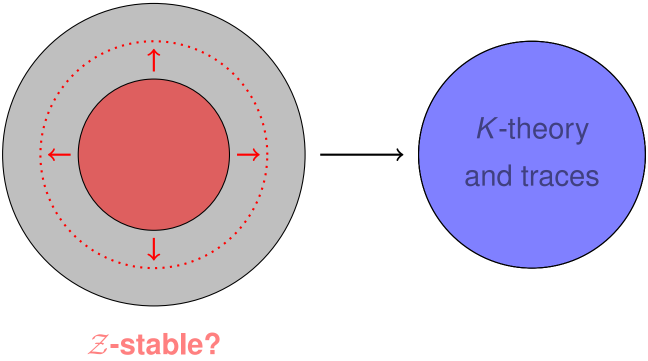

C^*-ALGEBRA

THE UNITAL CLASSIFICATION THEOREM

$\mathcal{Z}$-stable, simple, separable, unital, amenable C^*-algebras in the UCT class are classified by K-theory and traces.

- Analogue for C^*-algebras of the Murray-von Neumann, Connes, Haagerup classification of amenable von Neumann factors.
- These results 25+ year endeavour; work of many researchers.
- Dichotomy between traceless case (Kirchberg, Philips 94-00), and tracial case.

C^*-algebra: structure and classification

The classification theorem

$\mathcal{Z}$-stable, simple, separable, unital, amenable C^*-algebras in the UCT class are classified by K-theory and traces.

UCT class: satisfies a noncommutative universal coefficient theorem

- Computes Kasparov's bivariant KK-theory in terms of K-theory.
- $C(X)$ does satisfy the UCT; think of satisfying UCT as being homotopic (in a weak sense) to an abelian algebra.
- Major problem. Do all amenable C^*-algebras satisfy the UCT?
- But all amenable C^*-algebras which have been written down explicitly do.

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Range of invariant

Is understood: all possible K-theory trace pairings arise. Obtain structural consequences from classification:

- all classifiable C^*-algebras have twisted groupoid models
- Internal inductive limit structure arises from classification

Examples $C(X) \rtimes G$

The unital classification theorem

$\mathcal{Z}$-stable, simple, separable, unital, amenable C^*-algebras in the UCT class are classified by K-theory and traces.

For $C(X) \rtimes G$: Blue conditions easily described

- Unital: automatic
- Separability: from G countable discrete and X metrisable.
- Amenable: when G is amenable (or more generally precisely when the action is amenable)
- Simple: when action is topologically free and minimal.

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- UCT automatic when G (or action) is amenable.
- All that remains is $\mathcal{Z}$-stability - huge body of work in this direction.

(Roughly) how does it work? I

Classify maps $\phi, \psi : A \to B$

- Up to approximate unitary equivalence: there exist a sequence of unitaries (u_n) in B with $u_n \phi(a) u_n^* \to \psi(a)$ for all $a \in A$.

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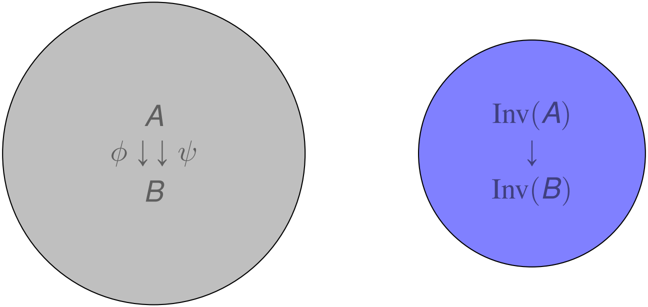

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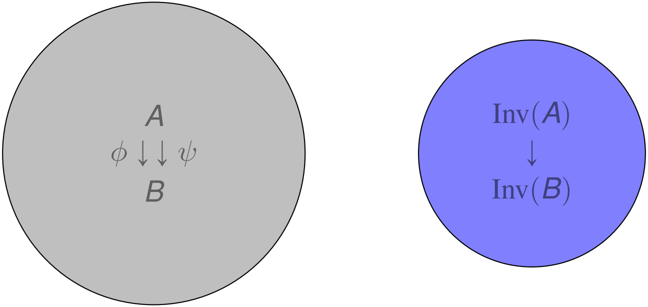

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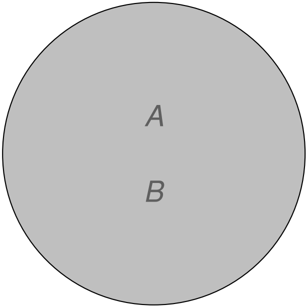

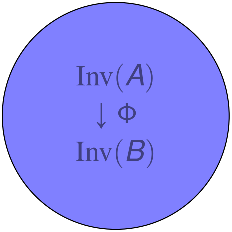

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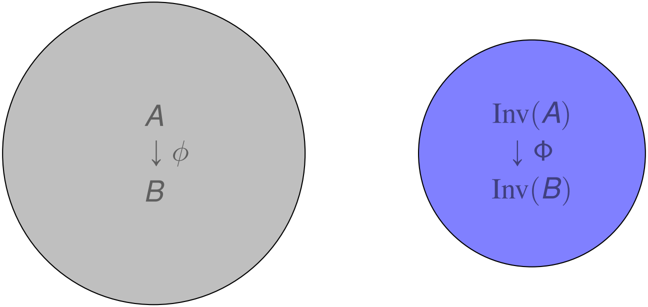

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But at first glance

It does not seem easier to produce a map $A \to B$ as compared to producing an isomorphism.

Doing things approximately is easier than doing them exactly

Classify approximately multiplicative maps $\phi, \psi : A \to B$

- Encode approximate multiplicativity using ultrapower. For a free ultrafilter $\omega \in \beta\mathbb{N}\backslash\mathbb{N}$, define

$$B_\omega = \ell^\infty(B)/\{(x_n) \in \ell^\infty(B) \colon \lim_{n\to\omega} \|x_n\| = 0\}.$$

- Then bounded sequences of *-linear maps $\phi_n : A \to B$ which are approximately multiplicative are encoded by a single *-homomorphism $\phi : A \to B_\omega$.

Easier to prove existence. But

Harder to prove uniqueness

How does it roughly work II:

Simplifying assumption: B has a unique trace τ

- gives 2-norm $\|x\|_2 = \tau(x^*x)^{1/2} \leqslant \|x\|$.

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$$A$$

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$$\begin{array}{ccc} & & A \\ & & \theta \big\downarrow \text{ classified by trace} \\ B_\omega & \longrightarrow & B^\omega \end{array}$$

Folklore consequence of von Neumann classification

Maps from separable nuclear C^*-algebra to a II_1 factor classified by the trace

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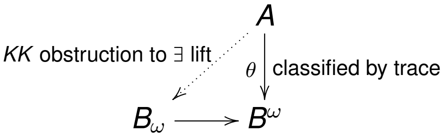

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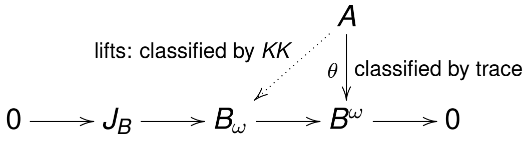

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$$\begin{array}{ccccccccc} & & & & & & A & & \\ & & & & & \swarrow^{\text{lifts: classified by } KK} & \Big\downarrow \theta \ \text{classified by trace} & & \\ 0 & \longrightarrow & J_B & \longrightarrow & B_\omega & \longrightarrow & B^\omega & \longrightarrow & 0 \end{array}$$

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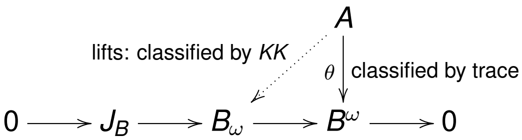

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As we need more data in the invariant for uniqueness. . .

. . . this means we have to prove a stronger existence theorem. . .

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Classification of tracial C^*-algebras obtained from lifting von Neumann classification and working with KK-theory.

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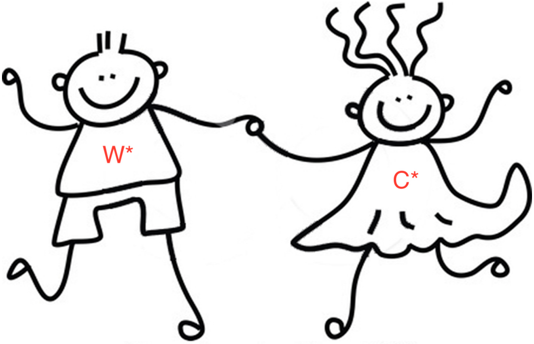

Thank you