Euler systems and the Bloch–Kato conjecture

David Loeffler
(UniDistance Suisse)

Mathematics Münster Mid-term Conference

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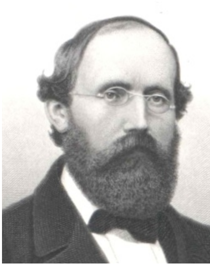
Riemann's zeta-function

- Zeta-function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \begin{pmatrix} s \in \mathbb{C}, \\ \mathrm{Re}(s) > 1 \end{pmatrix}$$



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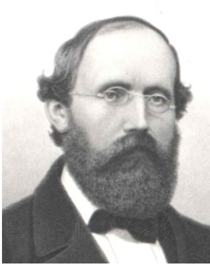

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- Riemann: use this & complex analysis to study **distribution of primes**

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Number fields

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- **Dedekind zeta function**:

$$\zeta_K(s) = \sum_{\mathfrak{a} \trianglelefteq \mathcal{O}_K} \frac{1}{\mathrm{Norm}(\mathfrak{a})^s} = \prod_{\substack{\mathfrak{p} \trianglelefteq \mathcal{O}_K \\ \text{prime ideal}}} (1 - \mathrm{Norm}(\mathfrak{p})^{-s})^{-1}$$

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Leading terms

Theorem (Analytic class number formula)

We have

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} R_K h_K}{w_K\sqrt{D_K}}$$

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- So the zeta-function (analytic object) *encodes* algebraic properties of K (class group / units)

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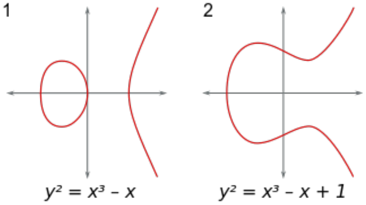
Function fields

- What other fields “behave like” algebraic number fields?



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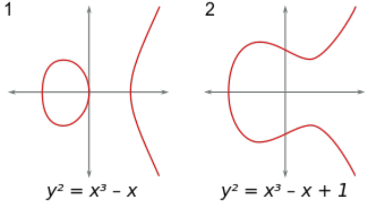
- Answer: **Function fields** of algebraic curves over finite fields, e.g.

$$y^2 = f(x), \qquad f \in \mathbb{F}_p[X]$$



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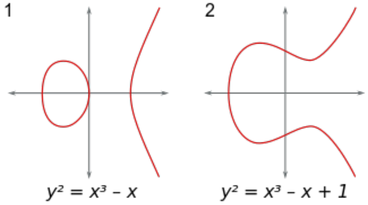
- Prime ideal $\mathfrak{p}$ for each point (x, y) of $\mathcal{C}$ (over $\mathbb{F}_p$ or any extension, up to Galois action)



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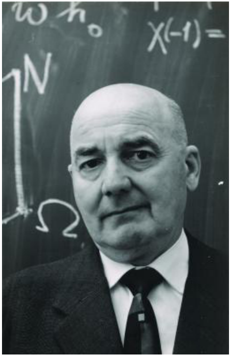

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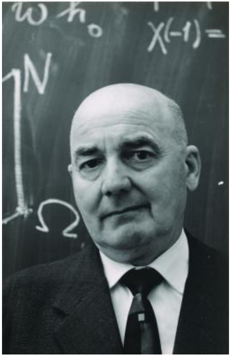

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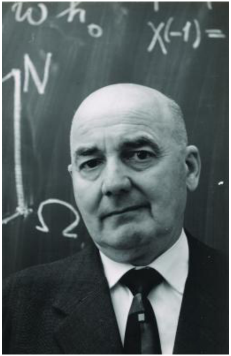
- Hasse, Weil: this is a *rational function* of p^{-s}, and satisfies an analogue of the Riemann hypothesis.



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Back to the rational numbers

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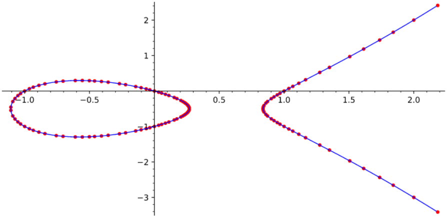

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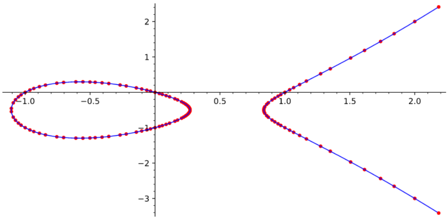

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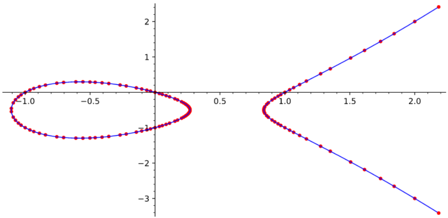

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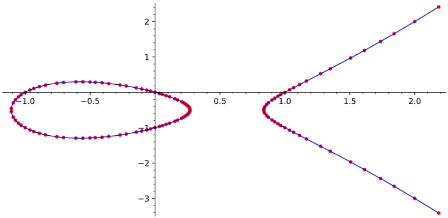

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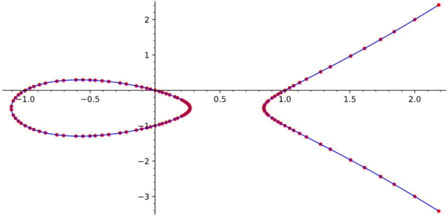
- Maybe some sort of generating function might explain this?



An outrageous idea

- Can reduce equations mod p (excluding finitely many bad primes)

$$E \rightsquigarrow E_p \quad \text{curve} / \mathbb{F}_p$$

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- Slight refinement:

$$L(E, s) := \frac{\zeta(s)\zeta(s-1)}{\prod_p \zeta_{E_p}(s)} \qquad \text{(removes some junk terms)}$$

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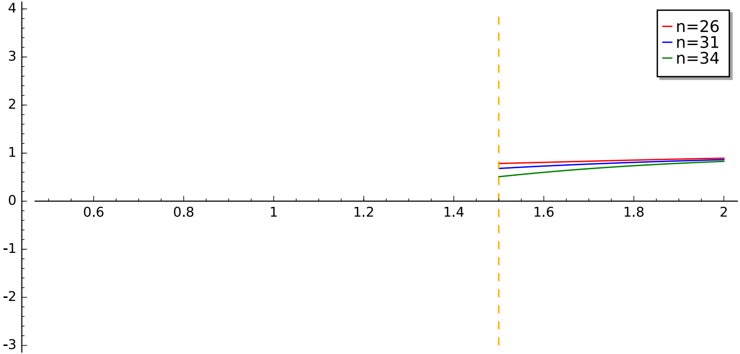
Some examples

$$\boxed{L(E, s) \text{ for } E := Y^2 = X^3 - n^2 X}$$



Rank of $E(\mathbb{Q})$: 0, 1, 2 respectively

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Analytic continuation

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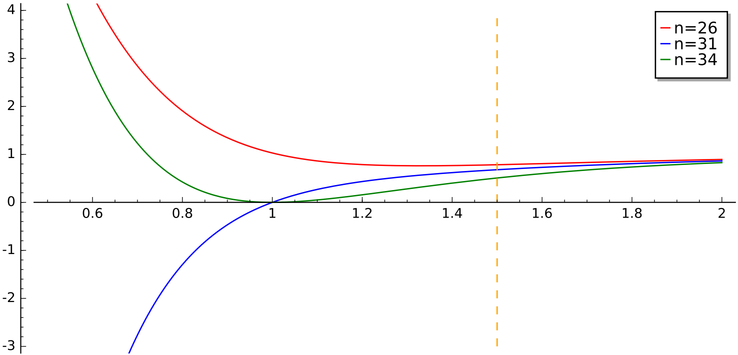

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The Birch–Swinnerton-Dyer conjecture

Conjecture (Birch–Swinnerton-Dyer, 1963)

Let E be an elliptic curve. Then: $\operatorname{ord}_{s=1} L(E, s) = \underbrace{r(E)}_{\text{rank of } E(\mathbb{Q})}$.

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- Also predict **leading term** at $s = 1$ in terms of finer algebraic invariants (regulator, Shafarevich–Tate group)

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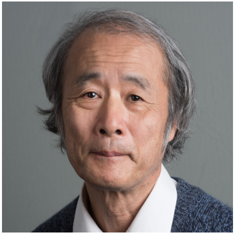
  - Doesn't make sense for general motives

The Bloch–Kato conjecture



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$$\operatorname{ord}_{s=n} L(M, s) \ =$$

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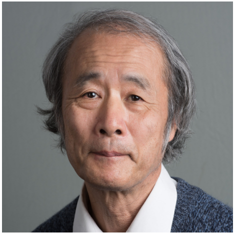

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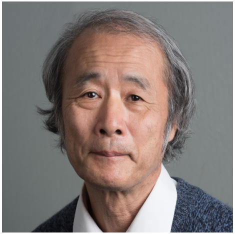

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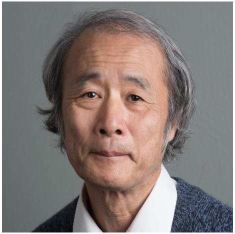

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- Refined form predicting leading term

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BSD for small orders of vanishing

Theorem (Kolyvagin, 1989)

Let $E/\mathbb{Q}$ be an elliptic curve. If $\operatorname{ord}_{s=1} L(E,s) = 0$ or 1, then $\operatorname{rank} E(\mathbb{Q}) = \operatorname{ord}_{s=1} L(E,s)$.

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- Still know virtually nothing for order of vanishing ≥ 2

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Modularity

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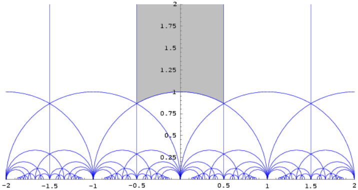

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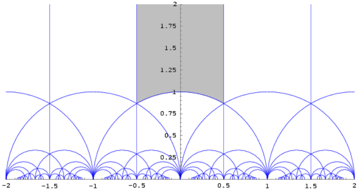
Modularity

- Upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$
- For $N \geq 1$ the group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - Nbc = 1 \right\}$$

acts on $\mathbb{H}$, and on compactification $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$



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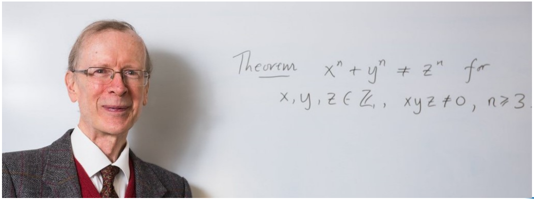
Modularity

- Say E is *modular* if for some N, $\exists$ complex-analytic map

$$\phi : \Gamma_0(N)\backslash\mathbb{H}^* \twoheadrightarrow E(\mathbb{C}).$$



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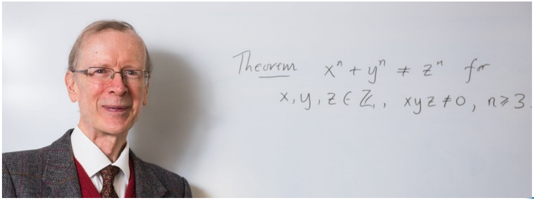

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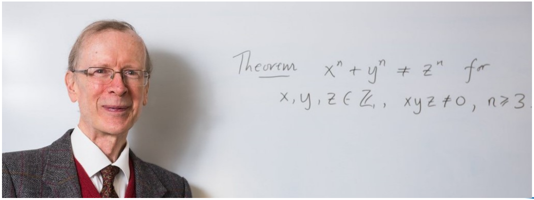
- Key to proof of Fermat's last theorem



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CM points and Heegner points

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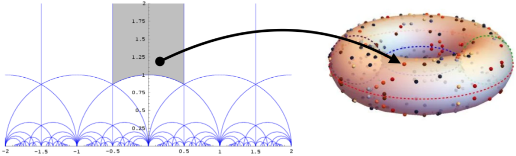
- **Heegner point** on a modular elliptic curve: image of a CM point under $\phi : \Gamma_0(N)\backslash\mathbb{H}^* \twoheadrightarrow E(\mathbb{C})$



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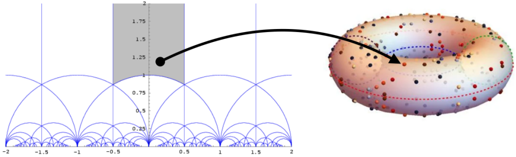

- Miracle: Heegner points are *algebraic*, i.e. lie in $E(\overline{\mathbb{Q}})$ (entirely un-obvious from construction)

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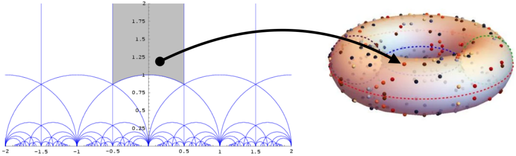
CM points and Heegner points

- A **CM point** is a point in $\mathbb{H}$ of form $a + \sqrt{-d}$, $a, d \in \mathbb{Q}$, $d > 0$
- **Heegner point** on a modular elliptic curve: image of a CM point under $\phi : \Gamma_0(N)\backslash\mathbb{H}^* \twoheadrightarrow E(\mathbb{C})$



- Miracle: Heegner points are *algebraic*, i.e. lie in $E(\overline{\mathbb{Q}})$ (entirely un-obvious from construction)
- *Shimura reciprocity* describes precisely which number field each one lives in (always an abelian extension of $\mathbb{Q}(\sqrt{-d})$)

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- Delicate manipulations with duality theory of Galois cohomology $\Rightarrow$ bounds on $E(\mathbb{Q})$: either it's zero, or c_1 generates it up to a finite error.

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- Proving non-triviality is more difficult (needs *explicit reciprocity laws*) – done for GSp_4, and for quadratic Hilbert modular groups

  [various works of Grossi, Lei, L., Pilloni, Skinner, Zerbes]

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- ... or something else? [Sangiovanni–Skinner, in preparation]

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