Goal: Compute $L(\mathcal{D}^p(R), \varphi^{gs}_R)$ for R Dedekind ring with Fraction field a number field (i.e. rings of integers like $R = \mathbb{Z}$). So far, we know

true for R Dedekind, $\mathrm{char}(K) \neq 2$ ($K = \mathrm{Frac}(R)$)

$$L_n(\mathcal{D}^p(R), \varphi^{gs}_R) \cong \begin{cases} L_n(\mathcal{D}^p(R), \varphi^q_R) & \text{for } n \leq -3 \\ L_n(\mathcal{D}^p(R), \varphi^s_R) & \text{for } n \geq -2 \end{cases}$$

Exercise: Show this for $n = -2, -1$.

The symmetric case

Let R be a Dedekind ring, S a collection of non-zero prime ideals of R

Definition: We let $R_S = \{x \in K \mid v_p(x) \geq 0 \text{ if } p \notin S\}$

$R_{(p)}$ is a discrete valuation ring, v_p the valuation extends to K.

Examples 1) $S = \{0 \neq p \subseteq R \text{ prime}\} \Rightarrow R_S = K$

2) If T is a multiplicative subset of R, $S = \{p \mid p \cap T \neq \emptyset\} \Rightarrow R_S = R[T^{-1}]$

3) If $S = \{p_1, \dots, p_n\}$ and $\exists r_i$ s.th. $p_1^{r_1} \cdots p_n^{r_n} = (x) \Rightarrow R_S = R[\frac{1}{x}]$

Proposition: the ext. of scalar functor $- \otimes_R R_S : \mathcal{D}(R) \longrightarrow \mathcal{D}(R_S)$ ext. to the projection of a Poincaré-Verdier sequence

$$(\mathcal{D}(R)_S, \varphi^s_M) \longrightarrow (\mathcal{D}(R), \varphi^s_M) \longrightarrow (\mathcal{D}(R_S), \varphi^s_{M_S})$$

indeed this is true for all $\varphi \geq m$ ($\varphi^{\geq m}_M$)

proof: • $M_S = M \otimes_R R_S \simeq M \otimes_{R \otimes R} (R_S \otimes R_S)$

• nat. comp. map: $\varphi^s_M(X) \longrightarrow \varphi^s_{M_S}(X \otimes_R R_S)$ given by

$$\mathrm{map}_{R \otimes R}(X \otimes X, M)^{hC_2} \xrightarrow{\text{can.}} \mathrm{map}_{R_S \otimes R_S}((X \otimes_R R_S) \otimes (X \otimes_R R_S), M \otimes_R R_S)$$

• $\mathcal{D}(R)_S$ = kernel of the proj. is generated by $\{R/p\}_{p \in S}$; $\mathcal{D}(R) \xrightarrow{f} \mathcal{D}(R_S)$ is a Verdier projection

a priori, up to idempotent completion, but the map f is ess. surjective.

$(K_0(R) \cong \mathbb{Z} \oplus \mathrm{Pic}(R) \to \mathbb{Z} \oplus \mathrm{Pic}(R_S) = K_0(R_S)$

$+ \; K^\times \longrightarrow \bigoplus_{0 \neq p \subseteq R \text{ prime}} \mathbb{Z} \twoheadrightarrow \mathrm{Pic}(R)$)

$x \in R \Rightarrow (r_i \mid p_i)$

$(x) = p_1^{r_1} \cdots p_n^{r_n}$

• $\varphi^s_{M_S}$ is left Kan extended from φ^s_M along $- \otimes_R R_S$.

the essential ingredient is that $M^{tC_2} \otimes_R R_S \simeq (M_S)^{tC_2}$

R_S = filtered colimit of free R-modules

the left Kan extension on linear + bilinear terms is just base-changed along $R \to R_S$.

Now we want to discribe $(\mathcal{D}^p(R)_S, \varphi^s_R)$ in terms of $(\mathcal{D}(R/p), ?)$ for $p \in S$.

So fix $p \in S$, and consider the restriction of scalars functor $\mathcal{D}^p(R/p) \to \mathcal{D}^p(R)$

Exercise: Show this is well-defined, i.e. that R/p is a perfect R-module.

Proposition: the restriction functor $\mathcal{D}^p(R/p) \xrightarrow{p^*} \mathcal{D}^p(R)$ refines to a Poincaré functor $(\mathcal{D}^p(R/p), \varphi^s_{R/p}) \xrightarrow{\text{dév}_\pi} (\mathcal{D}^p(R)_{\{p\}}, \Sigma\varphi^s_R)$ depending on a uniformiser π for p.

i.e. a generator of $p \subseteq R_{(p)}$ ← DVR ⇒ principle ideal domain.

$$(\mathcal{D}^p(R)_{\{p\}}, \Sigma\varphi^s_R) \cap (\mathcal{D}^p(R)_S, \Sigma\varphi^s_R)$$

Exercise: Show $R/p[-1] \simeq p_* R$ provided p is principal

proof: For the Poincaré Structure on p^* we calculate induced by the adj. of the equivalence $R/p[-1] \simeq p_* R$

$$\mathrm{map}_{R/p \otimes R/p}(X \otimes X, R/p) \xrightarrow{hC_2\, p^*} \mathrm{map}_{R\otimes R}(p^*(X\otimes X), p^* R/p) \xrightarrow[\otimes]{hC_2} \mathrm{map}_{R\otimes R}(p^*(X\otimes X), R[1])$$

Exercise: Show that the map $p^* R/p \to R[1]$ adj. to the above equiv. is the Bockstein as to π if p is principal.

$$R \xrightarrow{\pi} R \to R/p \xrightarrow{\beta_\pi} R[1]$$

Thm: Let R be a Dedekind ring, S set of non-zero primes with chosen uniformisers π_p. Then the map

$$\bigoplus_{p \in S} L(\mathcal{D}^p(R/p), \varphi^s_{R/p}) \xrightarrow{\text{dév}_{\pi_p}} L(\mathcal{D}^p(R)_S, \Sigma\varphi^s_R)$$

is an equivalence.

proof sketch: 1) reduce to the case $|S| < \infty$ (everything is comp. w/ filtered colimits).

2) $\prod_{p\in S} \mathcal{D}^p(R/p)$ has t-structure (productwise) s.th. $D(\mathcal{C}_{\leq 0}) \subseteq \mathcal{C}_{\geq 0}$ + $D(\mathcal{C}_{\geq 0}) \subseteq \mathcal{C}_{\leq 0}$

3) $\mathcal{D}^p(R)_S$ has t-structure (restricted from $\mathcal{D}^p(R)$) s.th. $D(\mathcal{C}_{\leq 0}) \subseteq \mathcal{C}_{\geq 0}$ + $D(\mathcal{C}_{\geq 0}) \subseteq \mathcal{C}_{\leq 0}$

4) Deduce that odd dim L-groups vanish + even dim ones are sym + anti sym. Witt groups of the heart of the t-structures

5) Use dévissage for (-1)-sym. Witt groups: $\bigoplus_{p\in S} W^{\pm s}(R/p) \xrightarrow{\cong} W^{\pm s}(\mathrm{Mod}(R)^{f.g.}_S)$ ∎

Corollary: R, S as before. There is a fibre sequence

$$L(\mathcal{D}^p(R), \varphi^s_R) \to L(\mathcal{D}^p(R_S), \varphi^s_{R_S}) \xrightarrow{\oplus \partial_\pi} \bigoplus_{p\in S} L(\mathcal{D}^p(R/p), \varphi^s_{R/p})$$

Example: $\partial_\pi: L_0(\mathcal{D}^p(K),\varphi^s_K) \longrightarrow L_0(\mathcal{D}^s(R/p),\varphi^s_{R/p})$ is given as follows

else replace R by $R_{(p)}$

$L_0(\mathcal{D}^p(K),\varphi^s_K)\cong W^s(K)$ generated by $\langle x\rangle$ with $x\in R\setminus\{0\}$. We may assume R is local so that $x=\pi^i\cdot u$ for $u\in R^\times$.

$u\in R/p$ via $R\to R/p$

suffices to specify $\partial_\pi(\langle u\rangle)=0$ and $\partial_\pi(\langle \pi u\rangle)=\langle u\rangle$

means that all residue fields are finite fields.

Corollary: R Dedekind, $K=\mathrm{Frac}(R)$ global field of characteristic 2. Then

$$L_n(\mathcal{D}^p(R),\varphi^s_R)\cong\begin{cases} W^s(R) & \text{if } n\equiv 0 \ (2)\\ \mathrm{Pic}(R)/2 & \text{if } n\equiv 1\ (2)\end{cases}$$

proof: there is a fibre sequence

$$L(\mathcal{D}^p(R),\varphi^s_R)\longrightarrow L(\mathcal{D}^p(K),\varphi^s_K)\xrightarrow{\oplus\partial_\pi}\bigoplus_{p\neq 0}L(\mathcal{D}^p(R/p),\varphi^s_{R/p})$$

fields have no odd L-groups

$$\begin{tikzcd}
\leadsto 0 \arrow[r] & L_0(\mathcal{D}^p(R),\varphi^s_R) \arrow[r] & L_0(\mathcal{D}^p(K),\varphi^s_K) \arrow[r] & \bigoplus L_0(\mathcal{D}^p(R/p),\varphi^s_{R/p}) \arrow[r] & L_{-1}(\mathcal{D}^p(R),\varphi^s_R) \arrow[r] \arrow[dd, squiggly, "\cong"] & 0 \\
 & W^s_0(R) \arrow[u, "\cong"] \arrow[r] & W^s_0(K) \arrow[u, "\cong"] \arrow[r, "\oplus\partial_\pi"] & \bigoplus_{p\neq 0} W^s(R/p) \arrow[u, "\cong"] & & \\
 & & \mathbb{Z}/2[K^\times] \arrow[u, dotted] \arrow[r] & \bigoplus_{p\neq 0}\mathbb{Z}/2 \arrow[u, "\cong\ (\text{Exercise } 5)"'] \arrow[r] & \mathrm{Pic}(R)/2 \arrow[r] & 0
\end{tikzcd}$$

∎

Fact: cokernel of $W^s(K)\xrightarrow{\partial_\pi}\bigoplus_{p\neq 0}W^s(R/p)$ is $\mathrm{Pic}(R)/2$ whenever K is global (eg. a number field)

Corollary: R Dedekind ring, K global field of char. $\neq 2$. Let $d=\#$ dyadic primes $(2\in p)$

$$L_n(\mathcal{D}^p(R),\varphi^s_R)\cong\begin{cases} W^s(R) & n\equiv 0\ (4)\\ (\mathbb{Z}/2)^d & n\equiv 1\ (4)\\ 0 & n\equiv 2\ (4)\\ \mathrm{Pic}(R)/2 & n\equiv 3\ (4)\end{cases}$$

proof: Exercise.

Corollary: $$L_n(\mathcal{D}^p(\mathbb{Z}),\varphi^s)=L^s_n(\mathbb{Z})=\begin{cases}\mathbb{Z} & n\equiv 0\ (4) & \text{inv.} = \text{signature}\\ \mathbb{Z}/2 & n\equiv 1\ (4) & \text{inv.} = \text{de Rham inv.}\\ 0 & n\equiv 2\ (4)\\ 0 & n\equiv 3\ (4)\end{cases}$$

The quadratic case

Exercise: Dévissage fails for $L(-;\varphi^q)$ maybe wait until the end of the lecture

else interpret $R^\wedge_x$ as the derived completion.

Thm: let R be a ring, $x\in R$ s.th x-power torsion is bounded. Then for $\varphi=\varphi^q,\varphi^s$

assume $K_0(R)\twoheadrightarrow K_0(R[\frac{1}{x}])$ + $K_0(R^\wedge_x)\twoheadrightarrow K_0(R^\wedge_x[\frac{1}{x}])$ } true for $R=$ Dedekind

$$\begin{tikzcd} L(\mathcal{D}^p(R),\varphi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L(\mathcal{D}^p(R[\frac{1}{x}]),\varphi) \arrow[d] \\ L(\mathcal{D}^p(R^\wedge_x),\varphi) \arrow[r] & L(\mathcal{D}^p(R^\wedge_x[\frac{1}{x}]),\varphi) \end{tikzcd}$$

is a pullback.

If $x\in\mathbb{Z}\to R$ $\Rightarrow$ same statement true for $\varphi^{\geq m}$ for all m.

proof:

$$\begin{tikzcd} (\mathcal{D}^p(R) \text{ on } x,\varphi) \arrow[r] \arrow[d, "\circledast"] & (\mathcal{D}^p(R),\varphi) \arrow[r, "\text{PV proj}"] \arrow[d, "\text{Poincaré functor}"] & (\mathcal{D}^p(R[\frac{1}{x}]),\varphi) \arrow[d] \\ (\mathcal{D}^p(R^\wedge_x) \text{ on } x,\varphi) \arrow[r] & (\mathcal{D}^p(R^\wedge_x),\varphi) \arrow[r, "\text{PV proj.}"] & (\mathcal{D}^p(R^\wedge_x[\frac{1}{x}]),\varphi) \end{tikzcd}$$

equivalence of categories compatible with duality.

φ^q,φ^s are determined by the duality, so $\circledast$ is an equiv. of Poincaré ∞-categories

applying $L(-)$ hence gives horizontal fibre sequences + fibres are equivalent. ∎

Thm (Wall) Let $I\subseteq R$ ideal, and assume R is I-adically complete. Then

$$L^q(R)=L(\mathcal{D}^p(R),\varphi^q_R)\xrightarrow{\simeq} L(\mathcal{D}^p(R/I),\varphi^q_{R/I})$$ is an equivalence.

Question: Is this true if $R\to R/I$ is only a Henselian pair?

Exercise: prove surjectivity on π_{2k} (this works more generally if $R\to S$ surjective + kernel $\subseteq$ Jacobson radical).

Corollary: Let R be a ring. Then there is a pullback

$$\begin{tikzcd} L^q(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L^q(R^\wedge_2) \simeq L^q(R/2) \arrow[d] \\ L^s(R) \arrow[r] & L^s(R^\wedge_2) \end{tikzcd}$$

Exercise: prove the Corollary.

Thm Let R be a Dedekind ring, K global, $\mathrm{char}(K) \neq 2$, $d = \#$ dyadic primes

$$L_n^q(R) \cong \begin{cases} W^q(R) & n \equiv 0 \ (4) \\ 0 & n \equiv 1 \ (4) \\ (\mathbb{Z}/2)^d & n \equiv 2 \ (4) \end{cases}$$

and $\exists$ an extension $0 \to A \to L_3^q(R) \to L_3^s(R) \to 0$

with $A = \mathrm{cokernel}\left(W^s(R) \oplus W^q(R_2^\wedge) \to W^s(R_2^\wedge) \right)$.

Exercise: prove the theorem. Hint: $W^q(\mathbb{F}_q) \cong \mathbb{Z}/2$ if $\mathbb{F}_q$ is a finite field of char. 2.

Corollary:

$$L^q(\mathbb{Z}) = \begin{cases} \mathbb{Z} & n \equiv 0 \ (4) \\ 0 & n \equiv 1 \ (4) \\ \mathbb{Z}/2 & n \equiv 2 \ (4) \\ 0 & n \equiv 3 \ (4) \end{cases}$$

signature divided by 8

Arf invariant

Exercise: signature of an even sym. form $/\mathbb{Z}$ is divisible by 8

proof: need $A = 0$: We have an exact sequence

$$\begin{tikzcd} W^q(\mathbb{Z}) \arrow[r, hook] \arrow[dr, bend right, "\text{Ex: } = 0"'] & W^s(\mathbb{Z}) \oplus W^q(\mathbb{Z}_2^\wedge) \arrow[r] \arrow[d] & W^s(\mathbb{Z}_2^\wedge) \arrow[r] & A \arrow[r] & 0 \\ & W^q(\mathbb{F}_2) & & & \end{tikzcd}$$

$\Rightarrow$ inj. map $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to W^s(\mathbb{Z}_2^\wedge)$ and by comp. to $W^s(\mathbb{Q}_2)$ one can see that $W^s(\mathbb{Z}_2^\wedge)$ has 16 elements. ∎

fundamental ideal is controlled by $H^*_{\acute{e}t}(\mathbb{Q}_2; \mu_2) = \begin{cases} \mathbb{Z}/2 \\ \mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2 \\ \mathbb{Z}/2 \end{cases}$

Exercise: If $d \geq 2 \Rightarrow A \neq 0$.