Determinacy, Partition Properties, and Combinatorics III

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Part III: Monotonicity and Continuity

We present two more recent results which are joint with W. Chan and N. Trang.

Suppose $\kappa \to (\kappa)^\lambda$ where $\lambda \leq \kappa$. We let μ_κ^λ be the measure on $[\kappa]_*^\lambda = \{f \colon \lambda \to \kappa$ of the correct type $\}$ defined by:

$\mu_\kappa^\lambda(A) = 1$ iff there is a c.u.b. $C \subseteq \kappa$ such that $[\kappa]_*^\lambda \subseteq A$.

Theorem (Chan, J, Trang)

Suppose $\kappa \to (\kappa)^\lambda$. Then the measure μ_κ^λ is monotonic. That is if $\Phi \colon [\kappa]_*^\lambda \to On$ then there is a c.u.b. $C \subseteq \kappa$ such that if $f, g \in [C]_*^\lambda$ and $f(\alpha) \leq g(\alpha)$ for all $\alpha < \lambda$, then $\Phi(f) \leq \Phi(g)$.

As a corollary we have the following.

Definition

We say a measure μ on δ is monotonic if for all $f : \delta \to \mathrm{On}$, there is a μ measure one set $A \subseteq \delta$ such that $f \restriction A$ in monotonically increasing.

If $\kappa \to (\kappa)^\lambda$ and ν is a measure on λ, we let $W(\nu)$, if $\lambda < \kappa$, or $S(\nu)$, if $\lambda = \kappa$, denote the measure on $j_\nu(\kappa)$ be the measure induced by μ_κ^λ and the measure ν.

Corollary

If $\kappa \to \kappa^\lambda$ and ν is a measure on λ, then the measure $W(\nu)$ or $S(\nu)$ on $j_\nu(\kappa)$ is monotonic.

We consider the case $\lambda = \kappa$.

Lemma

If $\Phi : [\kappa]^\kappa_* \to On$ then there is a c.u.b. $C \subseteq \kappa$ such that if $f, g \in [C]^\kappa_*$ and for all $\alpha < \kappa$ we have:

1. $f(\alpha) \leq g(\alpha)$
2. $g(\alpha) \neq \sup_{\beta<\alpha} f(\beta)$ *for all limit* $\beta < \kappa$.
3. $f(\alpha) \neq \sup_{\beta<\alpha} g(\beta)$ *for all limit* $\beta < \kappa$.

then $\Phi(f) \leq \Phi(g)$.

Fix $\mathcal{I}\colon \kappa \to \kappa$ increasing, discontinuous, with range in the additively indecomposable ordinals. We use functions of "indecomposable type $\mathcal{I}$."

For $h \in [\kappa]^\kappa_*$, let $\mathrm{main}(h)(\alpha) = h(\mathcal{I}(\alpha))$. Note that $\mathrm{main}(h)$ is also of the correct type.

$\mathcal{P}$: partition $h \in [\kappa]^\kappa_*$ according to whether

$$\forall p \in [h[\kappa]]^\kappa_* \; \Phi(\mathrm{main}(h)) \le \Phi(\mathrm{main}(p))$$

By wellfoundedness, on the homogeneous side of the partition the stated property holds. Fix C_0 homogeneous for $\mathcal{P}$ and let $C_1 \subseteq C_0$ be the closure points of C_0.

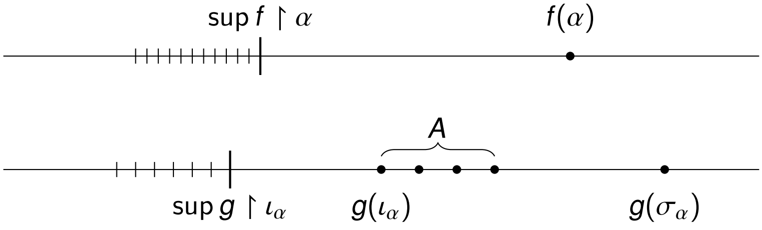
Fix $f, g \in [C_1]^\kappa_*$ satisfying (1)-(3), and we show that $\Phi(f) \leq \Phi(g)$.

We define two functions h, p with $h \in [C_0]^\kappa_*$, $p \in [h[\kappa]]^\kappa_*$, $\mathrm{main}(h) = f$, and $\mathrm{main}(p) = g$.

Let σ_β be least so that $g(\sigma_\beta) > f(\beta)$. So $\sigma_\beta \leq \beta + 1$.

Proceeding inductively on α we assume:

- For all $\beta < \alpha$, $h \restriction (\mathcal{I}(\beta) + 1)$ has been defined (of correct type) and $h(\mathcal{I}(\beta)) = f(\beta)$.
- For all $\beta < \alpha$, for all $\eta < \sigma_\beta$, $p \restriction (\mathcal{I}(\eta) + 1)$ has been defined and $p(\mathcal{I}(\eta)) = g(\eta)$.



- $\iota_\alpha = \sup_{\beta<\alpha} \sigma_\beta \leq \alpha$.
- $A = \{\beta \colon \sup f \restriction \alpha < g(\beta) < f(\alpha)\}$, o.t.$(A) = \xi$.
- If $A \neq \emptyset$ then $\iota_\alpha < \alpha$.
- $\sup f \restriction \alpha < g(\iota_\alpha)$ and $\sup(A) < f(\alpha)$ from our assumptions.

Let

$$\delta_0 = \sup\{\mathcal{I}(\beta) + 1 \colon \beta < \alpha\}$$
$$\tau_0 = \sup\{\mathcal{I}(\beta) + 1 \colon \beta < \iota_\alpha\}$$

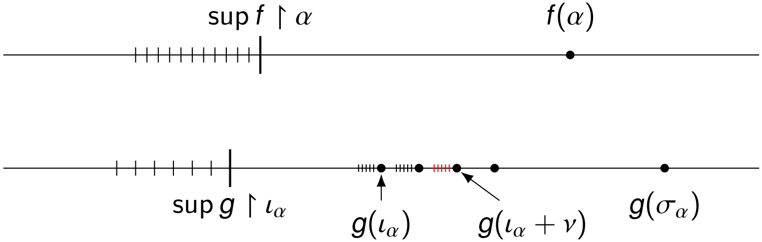
So we have defined $h \restriction \delta_0$, $p \restriction \tau_0$, and $\sup h \restriction \delta_0 = \sup f \restriction \alpha$, $\sup p \restriction \tau_0 = \sup g \restriction \iota_\alpha$.

For $\nu < \xi$, set:

$$\delta_\nu = \sup\{\delta_0 + \mathcal{I}(\iota_\alpha + \eta) + 1 \colon \eta < \nu\}$$
$$\epsilon_\nu = \delta_0 + \mathcal{I}(\iota_\alpha + \nu) = \delta_\nu + \mathcal{I}(\iota_\alpha + \nu)$$

$$\tau_\nu = \sup\{\tau_0 + \mathcal{I}(\iota_\alpha + \eta) + 1 \colon \eta < \nu\}$$
$$\mu_\nu = \tau_0 + \mathcal{I}(\iota_\alpha + \nu) = \tau_\nu + \mathcal{I}(\iota_\alpha + \nu)$$

Assume $h \restriction \delta_\nu$, $p \restriction \tau_\nu$ have been defined and $\sup h \restriction \delta_\nu = \sup p \restriction \tau_\nu = \sup g \restriction (\iota_\alpha + \nu)$.



For $\beta < \mathcal{I}(\iota_\alpha + \nu)$, define:

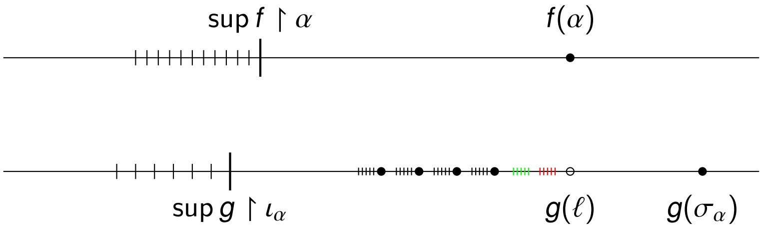
$$h(\delta_\nu + \beta) = p(\tau_\nu + \beta) = \mathbf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup h \restriction \delta\nu)$$

This defines $h \restriction \epsilon_\nu$, $p \restriction \mu_\nu$, and we then set $h(\epsilon_\nu) = p(\mu_\nu) = g(\iota_\alpha + \nu)$.

Let $\delta = \sup\{\epsilon_\nu + 1 \colon \nu < \xi\}$, $\tau = \sup\{\mu_\nu + 1 \colon \nu < \xi\}$.

So, $h \restriction \delta$, $p \restriction \tau$ have been defined and $\sup h \restriction \delta = \sup p \restriction \tau = \sup g \restriction (\iota_\alpha + \xi)$.

- Note that $\tau \le \delta \le \delta_0 + \sup(\mathcal{I} \restriction \alpha) + 1 < \delta_0 + \mathcal{I}(\alpha) = \mathcal{I}(\alpha)$.



Let $\ell = \min(\kappa \setminus A) = \iota_\alpha + \xi$. We could have $g(\ell) > f(\alpha)$ or $g(\ell) = f(\alpha)$.

If $g(\ell) > f(\alpha)$, set for $\beta < \mathcal{I}(\alpha)$: $h(\delta+\beta) = \textbf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup h \restriction \delta)$ and set $h(\mathcal{I}(\alpha)) = f(\alpha)$.

If $g(\ell) = f(\alpha)$ and $\ell = \alpha$, set for $\beta < \mathcal{I}(\alpha)$,

$h(\delta+\beta) = \textbf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup h \restriction \delta)$,

$p(\tau+\beta) = \textbf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup p \restriction \tau)$, If $g(\ell) = f(\alpha)$ and $\ell < \alpha$, set for $\beta < \mathcal{I}(\ell)$,

$$h(\delta + \beta) = \mathbf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup h \upharpoonright \delta),$$
$$p(\tau + \beta) = \mathbf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup p \upharpoonright \tau),$$

and for $\beta < \mathcal{I}(\alpha)$ set

$$h(\delta + \mathcal{I}(\ell) + \beta) = \mathbf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup h \upharpoonright \mathcal{I}(\ell)),$$

as shown.

Set $h(\mathcal{I}(\alpha)) = f(\alpha)$, $p(\mathcal{I}(\ell)) = g(\ell) = f(\alpha)$.

General Case

We now consider the general case, without the restrictions on f and g.

Let C_0 be homogeneous for the previous restricted version, and C_1 the closure points of C_0.

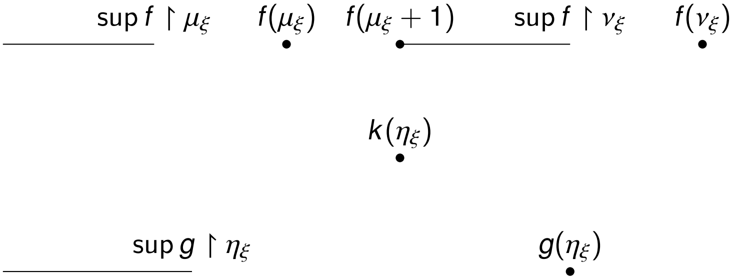
Fix $f, g \in [C_1]^\kappa_*$ with $f(\alpha) \leq g(\alpha)$ for all $\alpha < \kappa$.

- We first lower g to get k with $f \leq k \leq g$ and such that (k, g) satisfies the assumptions and (f, k) satisfies $k(\alpha)$ is not of the form $\sup f \restriction \beta$ for limit β.
- We then define h with $f \leq h \leq k$ where (h, k) and (f, h) satisfy the assumptions.

Definition of k:

Let (η_ξ, ν_ξ) enumerate the pairs with $g(\eta_\xi) = \sup f \restriction \nu_\xi$.

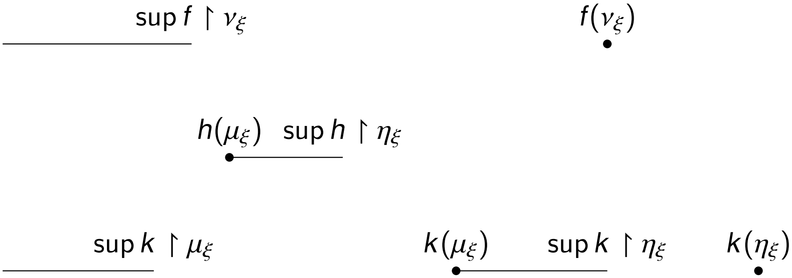
If α is not of the form η_ξ, let $k(\alpha) = g(\alpha)$.



We have $\eta_\xi \leq \mu_\xi < \mu_\xi + 1 < \nu_\xi$.

Definition of h:

Let (η_ξ, ν_ξ) enumerate the pairs with $f(\nu_\xi) = \sup k \restriction \eta_\xi$.



μ_ξ is least so that $k(\mu_\xi) > \sup f \restriction \nu_\xi$.

$\mu_\xi < \eta_\xi \leq \nu_\xi$.

Continuity and Application

Theorem (Chan, J, Trang)

Suppose $\epsilon < \kappa$, cof$(\epsilon) = \omega$, and $\kappa \to (\kappa)^{\epsilon \cdot \epsilon}$. Then for any $\Phi \colon [\kappa]^{\epsilon}_{*} \to On$, there is a c.u.b. $C \subseteq \kappa$ and a $\delta < \epsilon$ such that if $f, g \in [C]^{\epsilon}_{*}$ with $f \restriction \delta = g \restriction \delta$ and $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.

We have the following application.

Theorem (CJT)

Suppose $\kappa \to (\kappa)^{<\kappa}$. Then for all $\lambda < \kappa$, there does not exist an injection of $\kappa^{<\kappa}$ into ${}^{\lambda}On$.

We prove the application from the theorem.

Proof: Suppose $\Phi\colon \kappa^{<\kappa} \to {}^{\lambda}\mathrm{On}$ is injective.

For each $\gamma < \lambda$ and $\epsilon < \kappa$, Φ induces $\Phi^{\epsilon}_{\gamma}\colon [\kappa]^{\epsilon}_{*} \to \mathrm{On}$ by

$$\Phi^{\epsilon}_{\gamma}(f) = \Phi(f)(\gamma)$$

By the Theorem, $\forall \gamma < \lambda\ \forall \epsilon < \kappa\ \exists C \subseteq \kappa\ \exists \delta < \epsilon$ for all $f, g \in [C]^{\epsilon}_{*}$, if $\sup f = \sup g$ and $f \restriction \delta = g \restriction \delta$, then $\Phi^{\epsilon}_{\gamma}(f) = \Phi^{\epsilon}_{\gamma}(g)$.

Let $\delta^{\epsilon}_{\gamma} < \epsilon$ be the least such δ.

For each $\gamma < \lambda$, let $\delta_\gamma < \kappa$ be such that for almost all ϵ of cofinality ω, we have $\delta_\gamma^\epsilon = \delta_\gamma$.

Let $\delta^* = \sup_{\gamma<\lambda} \delta_\gamma < \kappa$.

For each $\gamma < \lambda$, there is an ω-club in κ of ϵ such that $\delta_\gamma^\epsilon = \delta_\gamma < \delta^*$.

By additivity of the club filter, we may fix an $\epsilon^* < \kappa$ so that for all $\gamma < \lambda$, $\delta_\gamma^{\epsilon^*} < \delta^*$.

So, for all $\gamma < \lambda$ there is a club $C \subseteq \kappa$ such that for all $f, g \in [C]^{\epsilon^*}_*$ with $\sup f = \sup g$ and $f \restriction \delta^* = g \restriction \delta^*$ we have $\Phi(f)(\gamma) = \Phi(g)(\gamma)$.

We need a variation of the additivity argument.

If we find a c.u.b. $C \subseteq \kappa$ that works for all $\gamma < \lambda$, then we have a contradiction:

Consider $f, g \in [C]^{\epsilon^*}_*$ with $\sup f = \sup g$, $f \restriction \delta^* = g \restriction \delta^*$ and with $f \neq g$.

Additivity argument.

For all $\gamma < \lambda$, $\forall^* f \in [\kappa]^{\epsilon^*}_*$ if $g \restriction \delta^* = f \restriction \delta^*$, and $g \sqsubseteq f$, then $\Phi(g)(\gamma) = \Phi(g)(\gamma)$.

By the additivity of the function space measure, there is a c.u.b. $C \subseteq \kappa$ such that $\forall \gamma < \lambda \; \forall f \in [C]^{\epsilon^*}_*$ if $g \restriction \delta^* = f \restriction \delta^*$, and $g \sqsubseteq f$, then $\Phi(f)(\gamma) = \Phi(g)(\gamma)$.

This c.u.b. $C \subseteq \kappa$ works. If $f, g \in [C]^{\epsilon^*}_*$, $f \restriction \delta^* = g \restriction \delta^*$, and $\sup(f) = \sup(g)$, then there is an $h \in [C]^{\epsilon^*}_*$ with $h \restriction \delta^* = f, g \restriction \delta^*$, and with $f \sqsubseteq h$, $g \sqsubseteq h$.

Proof of continuity

We first prove the following consequence of continuity.

Definition

We say $f, g \colon \epsilon \to \kappa$ are E_0-equivalent if $\exists \alpha \forall \beta \geq \alpha \ f(\beta) = g(\beta)$.

Theorem

Assume $\kappa \to \kappa^{<\kappa}$. Let $\epsilon < \kappa$ with cof$(\kappa) = \omega$. Let $\Phi \colon [\kappa]^{\epsilon}_{*} \to On$ be E_0-invariant. Then there is a c.u.b. $C \subseteq \kappa$ such that if $f, g \in [C]^{\epsilon}_{*}$ wirth $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.

Assume first that ϵ is indecomposable.

Partition: For $f \in [\kappa]^{\epsilon}_{*}$, $\mathcal{P}_1(f) = 1$ if there is a $g \sqsubseteq f$ with $\Phi(g) > \Phi(f)$.

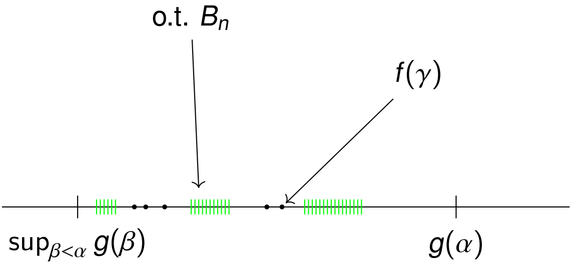
Assume towards a contradiction that $\mathcal{P}_1(f) = 1$ on the homogeneous side.

Partition: For $h \in [\kappa]^{\epsilon\cdot\epsilon}_{*}$, $\mathcal{P}_2(h) = 1$ iff letting $h' \colon \epsilon \to \kappa$ be $h'(\alpha) = \sup\{h(\epsilon \cdot \alpha + \beta) \colon \beta < \epsilon\}$, there is an $f \in [C_h]^{\epsilon}_{*}$ with $h' \sqsubseteq f$ and $\Phi(f) < \Phi(h')$.

From $\mathcal{P}_1 = 1$ on the homogeneous side we get $\mathcal{P}_2 = 1$ on the homogeneous side. Suppose not, and let C_1, C_2 be homogenous for $\mathcal{P}_1 = 1$ and $\mathcal{P}_2 = 0$. Let $C = (C_1 \cap C_2)'$.

Fix $f : \epsilon \to C$ of the correct type and $g \sqsubseteq f$ of correct type with $\Phi(g) > \Phi(f)$.

We construct $h \in [C]_*^{\epsilon \cdot \epsilon}$ with $h' = g$ and $h' \sqsubseteq f$, and $f \in [C_h]_*^{\epsilon}$. This contradicts $\mathcal{P}_2(h) = 0$.



Let $\rho\colon \omega \to \epsilon$ be cofinal. Let $b(\alpha)$ be least n such that $\rho(n) > \alpha$.
Let $B_n = \{\alpha\colon b(\alpha) = n\} = \rho(n) - \rho(n-1)$.

Let $r^\alpha\colon \epsilon \to (\sup_{\beta<\alpha} g(\beta), g(\alpha))$ with
$r^\alpha(\beta) = \textbf{next}_C^{\omega\cdot(\beta+1)}(G(\alpha, b(\beta)))$ where G witnesses g has uniform cofinality ω.

Let F^α be those points of $[sup_{\beta<\alpha} f(\beta), f(\alpha)) \cap \mathrm{ran}(f)$ not a limit of $\mathrm{ran}(r^\alpha)$.

Let $\{h(\epsilon \cdot \alpha + \beta) \colon \beta < \epsilon\}$ enumerate $\mathrm{ran}(r^\alpha) \cup F^\alpha$.

This shows that on the homogeneous side we have $\mathcal{P}_2 = 1$.

We now do an argument which contradicts $\mathcal{P}_1 = 1$ on the homogenous side.

Let $D_0 = (C_2)'$, and $D_{n+1} = D_n'$.

We construct functions $g_0, g_1, \dots$ in $[[D_1]^\epsilon_*$ with $\Phi(g_0) > \Phi(g_1) > \cdots$, a contradiction.

Let $g_0 \in [D_1]^\epsilon_*$ be such that $(*)$: for all n and all large enough $\alpha < \epsilon$ we have $g_0(\alpha) \in D_n$. Assume g_n also has these properties.

Let $h \colon \epsilon \cdot \epsilon \to D_0$ be such that if $\alpha \in D_m$, then $h(\epsilon \cdot \alpha + \beta) \in D_{m-1}$ and $h' = g_n$.

By homogeneity of C_2, there is a $\tilde{g}_{n+1} \in [C_h]^\epsilon_*$ with $g_n = h' \sqsubseteq \tilde{g}_{n+1}$ and $\Phi(g_{n+1}) < \Phi(g_n)$. Easily $\tilde{g}_{n+1}$ also satisfies $(*)$.

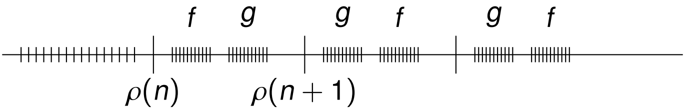
Let $g_{n+1} E_0 \tilde{g}_{n+1}$ and $g_{n+1} \in [D_1]^\epsilon_*$.

Now we prove the general continuity result.

Let $\Phi\colon [\kappa]^{\epsilon}_{*} \to \mathrm{On}$. We assume ϵ is indecomposible (general case similar). Again let $\rho\colon \omega \to \epsilon$ be cofinal.

We say (f, g) is of type n if $f, g\colon \epsilon \to \kappa$ are of the correct type and

1. $\forall \alpha < \rho(n)\ f(\alpha) = g(\alpha)$
2. $\sup f \restriction \rho(n) < g(\rho(n-1))$
3. For $m > n$, $\sup g \restriction \rho(m) < f(\rho(m-1))$



For each n we consider the partition

$\mathcal{P}^n$: partition pairs (f, g) of type n by:

$$\mathcal{P}^n(f,g) = \begin{cases} 0 & \text{if } \Phi(f) = \Phi(g) \\ 1 & \text{if } \Phi(f) < \Phi(g) \\ 2 & \text{if } \Phi(f) > \Phi(g) \end{cases}$$

Claim

There is an m^* such that for all $n \geq m^*$, $\mathcal{P}^n$ is homogeneous for the 0 *side.*

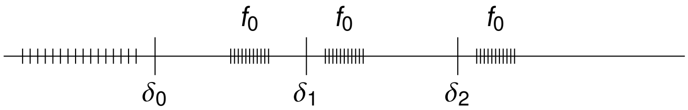
First suppose there are infinitely many n such that $\mathcal{P}^n$ is homogeneous for the 1 side.

Say $n_0 < n_1 < \cdots$ are homogeneous for the 1 side. Let C be a homogeneous set for the $\mathcal{P}^n$.

Fix $\delta_0 < \delta_1 < \cdots$ in C''.

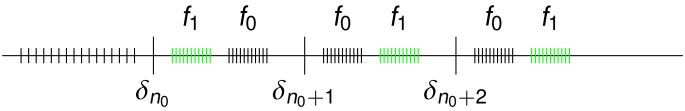
We define $f_0, f_1, \dots$ in $[C]^\epsilon_*$ such that (f_{i+1}, f_i) is of type n_i for all i. Then $\Phi(f_{i+1}) < \Phi(f_i)$ for all i, a contradiction.

Let $f_0 \in [C']^\epsilon_*$ with $f_0 \restriction [\rho(n-1), \rho(n)) \subseteq (\delta_{n-1}, \delta_n)$.



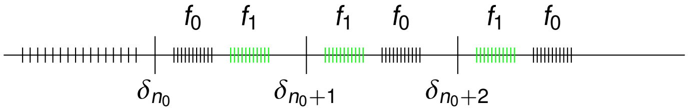
Given f_i, define f_{i+1} by:

- $f_{i+1} \restriction \rho(n_i) = f_i \restriction \rho(n_i)$
- For $\alpha \in [\rho(n_i), \rho(n_i+1))$, $f_{i+1}(\alpha) = \mathbf{next}_{C_0}^{\omega\cdot(\alpha+1)}(\delta_i)$
- For $j > n_i$ and $\alpha \in [\rho(j), \rho(j+1))$, $f_{i+1}(\alpha) = \mathbf{next}_{C_1}^{\omega\cdot(\alpha+1)}(\sup f_i \restriction \rho(j+1))$



Suppose there are infinitely many n for which $\mathcal{P}^n$ is homogeneous for the 2 side. Again fix $n_0 < n_1 < \cdots$.

The argument is similar to the previous case except now start with f_0 such that $f_0 \restriction [\rho(n-1), \rho(n)) \subseteq C^{(n)}$.



Fix $m^* \in \omega$ such that for all $n \geq m^*$, $\mathcal{P}^n$ is homogeneous for the 0 side. Let C be homogeneous for all of the $\mathcal{P}^n$. Let $\epsilon^* = \rho(m^*)$.

For $\ell \in [C]_*^{\rho(m*)}$, let $\Phi_\ell \colon [\kappa]_*^\epsilon \to \mathrm{On}$ be given by $\Phi_\ell(u) = \Phi(\ell^\frown u)$.

Claim

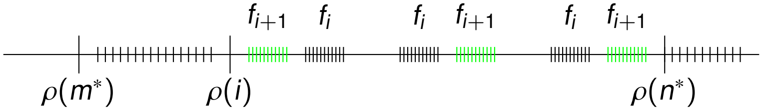
Φ_ℓ is E_0-invariant.

To see this, suppose $u, v \in [C'']_*^\epsilon$ with $u E_0 v$. Let $f = \ell^\frown u$, $g = \ell^\frown v$.

So, $f \restriction \rho(m^*) = g \restriction \rho(m^*)$, and say $\forall \alpha > \rho(n^*)\ f(\alpha) = g(\alpha)$.

Let $f_{m^*} = f$, $g_{m^*} = g$. We define $f_{m^*}, f_{m^*+1}, \dots, f_{n^*}$ and likewise for g with $f_{n^*} = g_{n^*}$, so that

$$\Phi(f) = \Phi(f_{m^*}) = \Phi(f_{m^*+1}) = \cdots = \Phi(f_{n^*})$$
$$= \Phi(g_{n^*}) = \cdots = \Phi(g_{m^*}) = \Phi(g)$$



- For $\alpha < \rho(i)$ $f_{i+1}(\alpha) = f_i(\alpha)$.
- For $\alpha \in [\rho(i), \rho(i+1))$, $f_{i+1}(\alpha) = \mathbf{next}_C^{\omega\cdot(\alpha+1)}(\sup f_i \restriction \rho(i))$
- For $j > i$ and $\alpha \in [\rho(j), \rho(j+1))$, $f_{i+1}(\alpha) = \mathbf{next}_{C_1}^{\omega\cdot(\alpha+1)}(\sup f_i \restriction \rho(j))$

So, for almost all ℓ, $\Phi_\ell(u)$ depends only on $\sup(u)$ for almost all u.

Let $\delta = \rho(m^*)$.

To finish, consider the partition:

$\mathcal{P}$: partition $f \in [\kappa]^\epsilon_*$ according to whether, if we let $\ell = f \restriction \delta$ and $u = f \restriction [\delta, \epsilon)$, for all $v \sqsubseteq u$ we have $\Phi(\ell^\frown u) = \Phi(\ell^\frown v)$.

Since $\Phi(\ell^\frown u)$ only depends on $\sup(u)$ almost everywhere, on the homogeneous side the stated property holds.

Let $C \subseteq \kappa$ be homogeneous for $\mathcal{P}$.

Then if $f, g \in [C]^\epsilon_*$, $f \restriction \delta = g \restriction \delta$, and $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$ (consider $f \cup g$).