Determinacy, Partition Properties, and Combinatorics I

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Part I: Introduction

We outline the theory and AD and AD^+ particularly as it relates to partition relations, combinatorics and definable cardinalities.

Some topics we will discuss include:

- Basic theory of AD, scales and Suslin cardinals.
- Paritition properties and introduction to analysis of measures.
- Computation of ultrapowers and uniform cofinalities.
- Recent consequences of partition properties such as monotonicity and continuity (joint with W. Chan and N. Trang).
- Applications to definable cardinalities in AD models.

We develop the basic theory assuming determinacy axioms.

Let X be a set. A game on X is a set $A \subseteq X^\omega$ which we view as the payoff of a two-player game:

$$\begin{array}{c|cccccccc} \mathrm{I} & x_0 & & x_2 & & x_4 & & \cdots & \\ \mathrm{II} & & x_1 & & x_3 & & x_5 & & \cdots \end{array}$$

I wins the run iff $x = (x_0, x_1, \dots) \in A$.

A strategy for I is a function $\sigma \colon \bigcup_n X^{2n} \to X$, and similarly for II.

If $x = (x_1, x_3, \dots) \in X^\omega$, let $\sigma * x = (x_0, x_1, \dots)$, where $x_{2n} = \sigma(x \restriction 2n)$. Similarly define $\tau * x$ if τ is a strategy for II.

We say the game on $A \subseteq X^\omega$ is determined if one of the players has a winning strategy.

AD_X is the assertion that every game on X is determined.

AD is the assertion that every game on $X = \omega$ is determined.

- AD was introduced by Mycielski and Steinhaus.
- AD is equivalent to AD_2.
- $\mathsf{AD}_{\mathbb{R}}$ is stronger than AD.
- $\mathsf{AD}_{\mathcal{P}(\mathbb{R})}$, AD_{ω_1} are inconsistent.

We generally work in the base theory $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$.

By Gale-Stewart, every open game $A \subseteq X^\omega$ (in the product of the discrete topologies) is (quasi) determined.

This follows from the rank-analysis of the game:

Let W_0 be the set of $s \in X^{<\omega}$ of even length such that $N_s \subseteq A$.

Let $W_{<\alpha} = \bigcup_{\beta<\alpha}$ for α limit.

Let $W_\alpha = W_{<\alpha} \cup \{s \colon \exists x \in X\ \forall y \in X\ (s^\frown x^\frown y \in W_{<\alpha})\}$

Let θ be least so that $W_\theta = W_{\theta+1}$. For $s \in W_\theta$, let $|s|$ be the least α such that $s \in W_\alpha$.

Then I has a winning (quasi) strategy from s if $s \in W_\theta$.

If $s \notin W_\theta$, then II has a winning (quasi) strategy from s. Namely, if I plays x, then II plays the (set of) y such that $s^\frown x^\frown y \notin W_\theta$.

This gives a canonical winning (quasi) strategy for a closed game.

Theorem (Martin)

(ZFC) Every Borel game on a set X is determined.

Hurkens and Neeman showed that in ZF, every Borel game is quasi-determined.

- (Harrington, Martin) $\boldsymbol{\Sigma}^1_1$-determinacy is equivalent to $\forall x\ x^{\#}$ exists.
- (Martin-Steel, Woodin) $\boldsymbol{\Sigma}^1_{n+1}$-determinacy is equiconsistent with $\exists n$ Woodin cardinals. $\boldsymbol{\Sigma}^1_{n+1}$ determinacy follows from $\exists n$ Woodin cardinals plus a measurable.
- (Woodin) $\mathrm{AD}^{L(\mathbb{R})}$ follows from $\exists \omega$ many Woodin cardinals plus a measurable. AD is eqiconsistent with $\exists \omega$ many Woodin cardinals.

Remark

Recently, Borel determinacy has found application to the theory of Borel equivalence relations.

For Γ a finitely generated group with a given presentation (a marked group), let $\chi_B(F(\omega^\Gamma))$ be the Borel chromatic number of the free part of the shift-action of Γ on the space ω^Γ.

For Γ, Δ countable groups, let $\Gamma * \Delta$ denote their free product.

Theorem (Marks)

$\chi_B(\omega^{\Gamma*\Delta}) \geq \chi_B(\omega^\Gamma) + \chi_B(\omega^\Delta) - 1.$

Theorem (Marks)

For each $2 \leq i \leq n+1$, there is an n-regular Borel graph with Borel chromatic number equal to i.

Definition

A tree on a set X is a set $T \subseteq X^{<\omega}$ closed under subsequence. $b \in X^\omega$ is a branch through T if $\forall n\, b \restriction n \in T$. We let $[T]$ denote the set of infinite branches through X.

Fact

A set $A \subseteq X^\omega$ is closed iff if there is a tree $T \subseteq X^{<\omega}$ such that $A = [T]$.

A Suslin representation generalizes this representation for closed sets.

Definition

If T is a tree on $X \times Y$, then $p[T] \subseteq X^\omega$ is defined by:

$$\begin{aligned} x \in p[T] &\text{ iff } \exists y \in Y^\omega\ (x, y) \in [T] \\ &\text{ iff } \exists y \in Y^\omega\ \forall n\ (x \restriction n, y \restriction n) \in T. \end{aligned}$$

Definition

$A \subseteq X^\omega$ is κ-Suslin if there is a tree T on $X \times \kappa$ such that $A = p[T]$. Let $S(\kappa)$ denote the collection of κ-Suslin subsets of ω^ω.

Fact

$S(\kappa)$ is a pointclass closed under $\exists^{\omega^\omega}$, countable unions and intersections and (Kechris), assuming AD*, non-selfdual.*

Definition

κ is a Suslin cardinal if $S(\kappa) \setminus \bigcup_{\lambda<\kappa} S(\lambda) \neq \emptyset$.

A Suslin representation of $A \subseteq \omega^\omega$ on κ is equivalent to a semi-scale on A into κ.

Definition

A semi-scale on A is a sequence of maps $\varphi_n \colon A \to \mathrm{On}$ such that if $x_m \in A$, $x_m \to x$, and for each n, $\varphi_n(x_m)$ is eventually constant, say equal to λ_n, then $x \in A$.

$\{\varphi_n\}$ is a scale on A if in addition, $\varphi_n(x) \leq \lambda_n$.

$\{\varphi_n\}$ is a $\boldsymbol{\Gamma}$-scale if the norm relations are in $\boldsymbol{\Gamma}$:

$$x <_n^* y \leftrightarrow (x \in A) \wedge [(y \notin A) \vee (y \in A \wedge \varphi_n(x) < \varphi_n(y))]$$
$$x \leq_n^* y \leftrightarrow (x \in A) \wedge [(y \notin A) \vee (y \in A \wedge \varphi_n(x) \leq \varphi_n(y))]$$

The Moschovakis periodicity theorems propagate the scale proprty under quantifiers.

Fact

(ZF) Assume scale$(\boldsymbol{\Gamma})$ where $\boldsymbol{\Gamma}$ is closed under $\forall^{\omega^\omega}$, $\wedge$, $\vee$. Then scale$(\exists^{\omega^\omega}\boldsymbol{\Gamma})$.

Theorem

($\boldsymbol{\Delta}$-det+$\mathrm{DC}_{\mathbb{R}}$) Assume scale$(\boldsymbol{\Gamma})$ where $\boldsymbol{\Gamma}$ is closed under $\exists^{\omega^\omega}$, $\wedge$, $\vee$. Then scale$(\forall^{\omega^\omega}\boldsymbol{\Gamma})$.

Corollary

(PD + $\mathrm{DC}_{\mathbb{R}}$) scale$(\boldsymbol{\Pi}^1_{2n+1})$, scale$(\boldsymbol{\Sigma}^1_{2n+2})$ for all $n \geq 0$.

We use the Erdös-Rado partition notation.

Definition

$\kappa \to (\lambda)^{\epsilon}_{\delta}$ if for every partition $\mathcal{P} \colon \kappa^{\epsilon} \to \delta$, there is a $H \subseteq \kappa$ with $|H| = \lambda$ such that $\mathcal{P} \restriction [H]^{\epsilon}$ is constant.

Remark

We usually have $\lambda = \kappa$.

We say κ has the strong partition property if $\kappa \to (\kappa)^{\kappa}_{2}$, and the very strong partition property if $\kappa \to (\kappa)^{\kappa}_{<\kappa}$. κ has the weak partition property if $\forall \epsilon < \kappa$ we have $\kappa \to (\kappa)^{\epsilon}_{2}$.

We abbreviate the strong and weak as $\kappa \to (\kappa)^{\kappa}$ and $\kappa \to (\kappa)^{<\kappa}$.

In the AD context an alternate form of the partition relations is preferred.

We say a function $f\colon \epsilon \to \kappa$ is of the correct type if it is increasing, discontinuous, and of uniform cofinality ω.

We let $[\kappa]^\epsilon_*$ denote the function from ϵ to κ of the correct type.

We say $\kappa \overset{\text{cub}}{\longrightarrow} \kappa^\epsilon$ if for every partition $\mathcal{P}\colon [\kappa]^\epsilon_* \to \{0,1\}$, there is a c.u.b. $C \subseteq \kappa$ such that $\mathcal{P} \restriction [C]^\epsilon_*$ is constant.

The two versions of the partition relation are essentially equivalent.

Fact

1. $\kappa \xrightarrow{cub} (\kappa)^{\epsilon}$ *implies* $\kappa \to (\kappa)^{\epsilon}$.
2. $\kappa \to (\kappa)^{\omega \cdot \epsilon}$ *implies* $\kappa \xrightarrow{cub} (\kappa)^{\epsilon}$.

In particular, the notion of weak and strong partition properties are the same for these two versions.

More generally, we have the c.u.b. version of the partition property for functions $\epsilon \to \kappa$ of any specified type, that is, any specified uniform cofinality.

Kunen Tree

Theorem

There is a tree T on $\omega \times \omega_1$ such that for all $f : \omega_1 \to \omega_1$ there is an $x \in \omega^\omega$ with T_x wellfounded and for all $\alpha \geq \omega$:

$$f(x) \leq |T_x \restriction \alpha|.$$

Proof: There is a tree W on $\omega \times \omega$ such that $\sup\{|W_x| : W_x \text{ is wellfounded }\} = \omega_1$.

Let $S \subseteq (\omega \times \omega_1)^{<\omega}$ be the tree of the nautural $\mathbf{\Pi}^1_1$-scale on WO.

Let T be the tree on $\omega \times \omega \times \omega_1 \times \omega \times \omega$ given by: $(s, t, \vec{\alpha}, u, v) \in T$ iff

1. $\exists \sigma, x, y$ extending s, t, u with $\sigma * x = y$.
2. $(t, \vec{\alpha}) \in S$.
3. $(u, v) \in W$.

To see this works, let $f : \omega_1 \to \omega_1$.

Play the Solovay game where I plays x, II plays y, and II wins iff

$$(x \in \mathrm{WO}) \to (W_y \text{ is wellfounded}) \wedge |W_y| > |x|)$$

By boundedness, II wins this game, say by σ.

Then for all $\alpha \geq \omega$, there is an $x \in \mathrm{WO}$ with $x \in p[S \restriction \alpha]$, and so $|W_{\sigma(x)}| > f(\alpha)$.

So, $|T_\sigma \restriction \alpha| > f(\alpha)$.

Uniform cofinalities at ω_1

We analyze the possible uniform cofinalities for a function $f\colon (\omega_1)^n \to \omega_1$.

By the partition relation $\omega_1 \to (\omega_1)^{n+1}$, there is a function $g\colon \omega_1 \to \omega_1$ such that $\forall^* \alpha_1 < \cdots < \alpha_n \; f(\alpha_1, \ldots, \alpha_n) < g(\alpha_n)$.

Let $x \in \omega^\omega$ be such that

$$\forall^* \vec{\alpha} \; f(\vec{\alpha}) < g(\alpha_n) < |T_x \restriction \alpha_n|.$$

Let $h(\vec{\alpha}) \leq \alpha_n$ be least so that for some function $\ell\colon \{(\vec{\alpha}, \beta) \colon \beta < h(\vec{\alpha})\} \to \omega_1$ we have, for almost all $\vec{\alpha}$:

$$\sup\{\ell(\vec{\alpha}, \beta) \colon \beta < \vec{h}(\alpha)\} = f(\vec{\alpha}).$$

Claim

$\forall^* \alpha_1, \ldots, \alpha_n \; h(\vec{\alpha}) = \alpha_i$ for some i, or $h(\vec{\alpha})$ is almost everywhere constant.

Suppose $\forall^* \vec{\alpha} \; \alpha_i < h(\vec{\alpha}) < \alpha_{i+1}$. Let $h'(\vec{\alpha}) = \alpha_i$.

By a partition as above, there is a function $k \colon \omega_1 \to \omega_1$ such that $\forall^* \vec{\alpha} \; h(\vec{\alpha}) < k(\alpha_i)$.

Fix y so that $k(\beta) < |T_y \restriction \beta|$ almost everywhere.

Define $\ell'(\vec{\alpha}, \beta)$ for $\beta < \alpha_i$ by

$$\ell'(\vec{\alpha}, \beta) = \ell(|T_y \restriction \alpha_i(\beta)|)$$

if $|T_y \restriction \alpha_i(\beta)| < h(\vec{\alpha})$, and 0 otherwise.

Then h', ℓ' violates the minimality of h, ℓ.

So either $h(\vec{\alpha}) = \alpha_i$ or $h(\vec{\alpha})$ is constant almost everywhere.

In the first case we have that $f(\vec{\alpha})$ has uniform cofinality α_i almost everywhere. In the second case, $f(\vec{\alpha})$ has uniform cofinality ω almost everywhere.

Fact

These uniform cofinalities are distinct.

A similar analysis describes the (almost everywhere) type of an arbitrary $f\colon \omega_1^n \to \omega_1$.

There is a partial permutation $\pi = (i_1, \dots, i_k)$ of $(1, \dots, n)$ beginning with n so that $f(\vec{\alpha}) < f(\vec{\beta})$ iff

$$(\alpha_{i_1}, \dots, \alpha_{i_k}) <_{\text{lex}} (\beta_{i_1}, \dots, \beta_{i_k}).$$

Then either:

- $f(\vec{\alpha})$ has uniform cofinality ω.
- $f(\vec{\alpha})$ is continuous almost everywhere.
- There is a partial permutation π' extending π which gives the unform cofinality.

Fact

Assuming AD, *every ultrafilter on a set X is countably additive.*

The (cub) partition relation $\kappa \to (\kappa)^2$ gives that the ω-cofinal c.u.b. filter of κ is a normal measure W^1_1 on κ.

Let W^n_1 denote the n-fold product of W^1_1.

Theorem ($\mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$)

Let μ be a measure on ω_1. Then μ is equivalent to W^n_1 for some n (or to a principal measure).

Proof: Assume μ is non-principal.

Let $f_1 : \omega_1 \to \omega_1$ represent the least equivalence class such that f_1 is almost everywhere non-constant, and monotonically increasing.

Let $\nu_1 = f_1(\mu)$. Then $\nu_1 = W^1_1$. Fix a μ measure one set A_1 on which f_1 is monotonically increasing.

Let $g_1(\beta) = \sup\{\alpha \in A_1 : f_1(\alpha) \leq \alpha\}$.

Let x_1 be such that $\forall^* \beta_1 \; g_1(\beta_1) < |T_{x_1} \restriction \beta_1|$.

For μ almost all α, let $r_1(\alpha)$ be such that

$$\alpha = |T_{x_1} \restriction f_1(\alpha)(r_1(\alpha))|.$$

Now we proceed with the measure $r_1(\mu)$.

Consider the case r_1 not constant almost everywhere. Note that a.e. $r_1(\alpha) < f_1(\alpha)$.

Let f_2 represent the least μ equivalence class such that f_2 is not a.e. constant, and is a.e. monotonically increasing with respect to r_1.

That is, there is a μ measure one set A such that if α, α' are in A, $f_1(\alpha) = f_1(\alpha')$, and $r_1(\alpha) \leq r_1(\alpha')$, then $f_2(\alpha) \leq f_2(\alpha')$.

Note that there does not exist a c.u.b. $C \subseteq \omega_1$ and a μ measure one set A such that for all $\beta \in C$, $\{f_2(\alpha) : f_1(\alpha) = \beta \wedge \alpha \in A\}$ is bounded below $f_1(\alpha)$. [Otherwise r_2 is constant μ almost everywhere.]

Claim

We have $f_2(\mu) = W_1^1$.

For suppose $C \subseteq \omega_1$ is c.u.b. and $\forall^*_\mu \alpha\ f_2(\alpha) \notin C$.

Let $f_2' = \ell_C \circ f_2$ where $\ell(\gamma)$ is the largest element of $C \geq \gamma$.

Then for μ almost all α we have $f_2'(\alpha) < f_2(\alpha)$ and f_2' is monotonically increasing "on the f_1 blocks" with respect to r_1. Also, f_2' is not constant μ almost everywhere. This contradicts the definition of f_2.

Fix a μ measure one set $A_2 \subseteq A_1$ on which f_2 is monotonically increasing on the f_1 blocks with respect to r_1.

Define g_2 by:

$$g_2(\beta_2, \beta_1) = \sup\{r_1(\alpha) \colon \alpha \in A_2 \wedge f_1(\alpha) = \beta_1 \wedge f_2(\alpha) = \beta_2\}.$$

Then for W_1^2 almost all (β_2, β_1), $g_2(\beta_2) < \beta_1$. This follows from the monotonicity of f_2 on the f_1 blocks and the fact that f_2 is not constant μ almost everywhere.

For W_1^2 almost all (β_2, β_1), $g_2(\beta_2, \beta_1)$ depends only on β_2.

Fix x_2 such that for W_1^1 almost all β_2, $g_2(\beta_2) < |T_{x_2} \restriction \beta_2|$.

This then defines r_2: $\forall^*_\mu \alpha$

$$\alpha = |T_{x_1} \restriction f_1(\alpha)(|T_{x_2} \restriction f_2(\alpha)(r_2(\alpha))|)|$$

Continuing, we define $f_1, \dots, f_n$, $g_1, \dots, g_n$ for some n, reals $x_1, \dots, x_n$, and $r_1, \dots, r_n$ such that r_n is constant almost everywhere, say equal to δ.

We then have: $\forall^*_\mu \alpha$

$$\alpha = |T_{x_1}(f_1(\alpha))(|T_{x_2}(f_2(\alpha))(\cdots(|T_{x_n}(f_n(\alpha)(\delta))|)\cdots)|)|$$

We also have that if $F(\alpha) = (f_1(\alpha), \ldots, f_n(\alpha)$, then $F(\mu) = W^n_1$.

Let $G(\beta_1, \ldots, \beta_n) = |T_{x_n} \restriction (\beta_n)(G_{n-1}(\beta_1, \ldots, \beta_{n-1}))|$, where
$G_k(\beta_1, \ldots, \beta_k) = |T_{x_k} \restriction \beta_k(G_{k-1}(\beta_1, \ldots, \beta_{k-1}))|$,

and $G_0(\emptyset) = \delta$.

We have defined a μ measure one set A_n on which F is one-to-one and $F(\mu) = W_1^n$.

This completes the analysis of measures on ω_1.

Proving partition relations

We present the general framework, due to Martin for proving partition relations from AD.

Definition

Let $\lambda \leq \kappa$, where $\lambda \in \text{On}$, κ a cardinal. We say κ is λ-reasonable if there is a non-selfdual pointclass $\boldsymbol{\Gamma}$ closed under $\exists^{\omega^\omega}$ and a map ϕ with domain ω^ω satisfying:

1. $\phi(x) \subseteq \lambda \times \kappa$.
2. $\forall f \colon \lambda \to \kappa \; \exists x \in \omega^\omega \; \phi(x) = f$.
3. $\forall \alpha < \lambda \; \forall \beta < \kappa \; R_{\alpha,\beta} \in \boldsymbol{\Delta}$, where $x \in R_{\alpha,\beta} \leftrightarrow \phi(x)(\alpha,\beta) \wedge (\phi(x)(\alpha,\beta') \rightarrow \beta' = \beta)$.
4. Suppose $\alpha < \lambda$, $A \in \exists^{\omega^\omega} \boldsymbol{\Delta}$, and $A \subseteq R_\alpha = \{x \colon \exists \beta < \kappa \; x \in R_{\alpha,\beta}\}$. Then $\exists \beta_0 < \kappa \; \forall x \in A \; \exists \beta < \beta_0 \; \phi(x)(\alpha,\beta)$.

Theorem (Martin)

Suppose κ is $\omega \cdot \lambda$ reasonable. Then $\kappa \to \kappa^\lambda$.

Proof: Assume that $\mathbf{\Delta}$ is closed under $< \kappa$ unions and intersections (this actually follows).

Let $\mathcal{P} \colon \kappa^\lambda_* \to \{0, 1\}$ partition the functions of the correct type.

Play the game: I plays out x, II plays out y.

- If there is a least $\alpha < \omega \cdot \lambda$ such that $\neg R_\alpha(x)$ or $\neg R_\alpha(y)$, then I wins iff $R_\alpha(x)$.
- Otherwise, let f_x, f_y be the functions they determine: $f_x(\alpha) = \beta$ iff $R_{\alpha,\beta}(x)$. Let

$$f_{x,y}(\alpha) = \sup\{\max(f_x(\alpha'), f_y(\alpha')) \colon \alpha' < \omega \cdot (\alpha + 1)\}.$$

Then II wins iff $\mathcal{P}(f_{x,y}) = 1$.

Say II has a winning strategy τ.

Define a c.u.b. $C \subseteq \kappa$ as follows.

For $\alpha < \omega \cdot \lambda$, $\beta < \kappa$, let

$$x \in S_{\alpha,\beta} \leftrightarrow \forall \alpha' \leq \alpha \; \exists \beta' \leq \beta \; R_{\alpha',\beta'}(x).$$

So, $S_{\alpha,\beta} \in \mathbf{\Delta}$. So $\tau[S_{\alpha,\beta}] \in \exists^{\omega^\omega} \mathbf{\Delta}$.

Also, $\tau[S_{\alpha,\beta}] \subseteq R_\alpha$.

Let $g(\alpha,\beta) = \sup\{\phi(x)(\alpha) \colon x \in \tau[S_{\alpha,\beta}]\} < \kappa$.

Let $C \subseteq \kappa$ be closed under g.

Then C is homogeneous for $\mathcal{P}$:

- Let $f\colon \lambda \to C'$ be of the correct type.
- Let x be such that $\phi(x)$ codes a function f_x (i.e., $x \in R_\alpha$ for all $\alpha < \omega \cdot \lambda$) and f_x induces f (i.e., $f(\alpha) = \sup\{f_x(\alpha')\colon \alpha' < \omega \cdot \alpha\}$).
- Let $y = \tau(x)$, so y codes $f_y\colon \omega \cdot \lambda \to \kappa$.
- For all α, $f_y(\omega \cdot \alpha + n) < f_x(\omega \cdot \alpha + n + 1)$, so $f_{x,y} = f$.
- Since τ is winning for II, $\mathcal{P}(f) = 1$.

Fact (AD)

Every ultrafilter on a set X is countably additive (i.e., a measure).

Fact (AD)

(Martin) The cone filter is a measure on the set $\mathcal{D}$ of Turing degrees.

Definition

Θ is the supremum of the lengths of the pwos of $\mathbb{R}$.

Fact

(Kunen) Let $\lambda < \Theta$. Then every countably additive filter $\mathcal{F}$ on λ can be extended to a measure on λ.

Proof: Let $\pi\colon \omega^\omega \to \mathcal{P}(\lambda)$ be onto (coding lemma).

Let ν be the Martin measure on $\mathcal{D}$.

For $d \in \mathcal{D}$, let

$$f(d) = \min \bigcap\{\pi(x) \colon x \in d \wedge \pi(x) \in \mathcal{F}\}.$$

Let $\mu = f(\nu)$.

Fix the Kunen tree T at ω_1.

We say $\tau \in \omega^\omega$ is a code for a c.u.b. set if $\forall x \in \mathrm{WO}\ \tau(x) \in \mathrm{WO}$.

Let $C_x = \{\alpha < \omega_1 : \forall \gamma < \alpha\ |T_x \restriction \gamma| < \alpha\}$.

Fact

For every c.u.b. $C \subseteq \omega_1$ there is a code x such that $C_x \subseteq C$.

Definition

A set $S \subseteq \omega_1$ is simple if there is a c.u.b. code τ, an $\alpha_0 < \omega_1$, $x_1, \dots, x_n$ with T_{x_i} wellfounded such that

$$S = \{\alpha \colon \exists \alpha_1 < \cdots < \alpha_n \in C_\tau \ \alpha = h_n(\alpha_1, \dots, \alpha_n; \vec{x})\}$$

where

$$h_i(\alpha_1, \dots, \alpha_i; \vec{x}) = |T_{x_i} \restriction \alpha_i(h_{i-1}(\alpha_1, \dots, \alpha_{i-1}; \vec{x}))|$$

and

$$h_0(\vec{x}) = \alpha_0.$$

A code for the simple set S a real of the form $(x_0; x_1, \dots, x_n; \tau)$ where τ is a c.u.b. code, $x_0 \in \mathrm{WO}$, and T_{x_i} are wellfounded.

Following an argument of Kunen we show:

Fact

Every $A \subseteq \omega_1$ is a countable union of simple sets.

Proof: Let $\mathcal{I}$ be the σ-ideal generated by the simple sets contained in A.

Assume toward a contradiction $\mathcal{I}$ is a proper ideal, and let μ be a measure on A extending the corresponding filter $\mathcal{F}$.

By the analysis of measures on ω_1, there are $x_1, \dots, x_n$ with T_{x_i} wellfounded and an $\alpha_0 < \omega_1$ such that for all $B \subseteq \omega_1$ (assuming B is not bounded):

$$\mu(B) = 1 \leftrightarrow \exists \text{c.u.b. } C \subseteq \omega_1 \forall \beta_1 < \cdots < \beta_n \in C \\ h_n(\alpha_0; \beta_1, \dots, \beta_n, x_1, \dots, x_n) \in B.$$

Since $\mu(A) = 1$, we may fix a c.u.b. code τ, a $x_0 \in \text{WO}$ coding α_0, and the $x_1, \dots, x_n$ above.

Let $S = S(x_0; x_1, \dots, x_n; \tau)$ be the simple set given by these reals, so $S \subseteq A$.

Then $\mu(S) = 1$, but this contradicts $S \in \mathcal{I}$.

We now define the coding map ϕ. As a warm-up we first define a coding for subsets of ω_1, so $\phi(x) \subseteq \omega_1$.

View $x \in \omega^\omega$ as coding countably many $(x_0^i; x_1^i, \dots, x_{n_i}^i; \tau^i)$.

Set $\phi(x)(\alpha)$ iff $\exists i\ \alpha \in S(x^i) = S(x_0^i; x_1^i, \dots, x_{n_i}^i; \tau^i)$ iff

$$\exists i\ \exists \beta_1 < \cdots < \beta_{n_i} \in C_{\tau^i} \cap \alpha\ [|x_0^i| < \alpha \\ \wedge h(|x_0^i|; \beta_1, \dots, \beta_{n_i}; x_1^i, \dots, x_{n_i}^i) = \beta].$$

So, every $A \subseteq \omega_1$ is of the form $\phi(x)$ for some $x \in \omega^\omega$.

This is a $\boldsymbol{\Delta}^1_1$-coding of the subsets of ω_1:

For all $\alpha < \omega_1$, $\{x : \phi(x)(\alpha)\} \in \boldsymbol{\Delta}^1_1$.

We modify this coding to code functions from ω_1 to ω_1. So, $\phi(x) \subseteq \omega_1 \times \omega_1$.

It is not quite good enough to just regard $f : \omega_1 \to \omega_1$ as a subset of $\omega_1 \times \omega_1 \approx \omega_1$.

Suppose $f : \omega_1 \to \omega_1$ is increasing.

A simple subfunction $S \subseteq f$ is one where there is a c.u.b.code τ, $x_1, \dots, x_n$ with T_{x_i} wellfounded, and two $\gamma_0, \gamma_1 < \omega_1$ such that:

$$(\alpha,\beta) \in S \leftrightarrow \exists \max\{\gamma_0, \gamma_1\} < \beta_1 < \cdots < \beta_n < \alpha$$
$$[\beta_1, \dots, \beta_n \in C_\tau \wedge h(\gamma_0, \beta_1, \dots, \beta_n; \vec{x}) = \alpha$$
$$\wedge\, h(\gamma_1, \beta_1, \dots, \beta_n; \vec{x}) = \beta]$$

An argument similar to that for sets shows that every function f is a countable union of simple subfunctions.

- We let $X = f$, and analyze the measures on X.
- If μ is a measure on X, let $f_0 \colon X \to \omega_1$ represent the least equivalence class of a function which is not μ a.e. constant and monotonically increasing in the first argument (if $\alpha_1 \leq \alpha_2$, then $f_0(\alpha_1, \beta_1) \leq f_0(\alpha_2, \beta_2)$).
- $f_0(\mu) = W^1_1$ as before.
- Let $g_0(\delta) = \sup\{\max\{\alpha, \beta\} \colon (\alpha, \beta) \in X \wedge f_0(\alpha) \leq \delta\}$.
- The rest of the argument proceeds as before.

Weak partition relation at ω_2

Fix $\lambda < \omega_2$, and we show $\omega_2 \to (\omega_2)^\lambda$.

Fix a function $h\colon \omega_1 \to \omega_1$ with $[h]_{W_1^1} = \lambda$.

Say a function f is of type h if $\mathrm{dom}(f) = \{(\alpha,\beta)\colon \alpha < h(\beta)\}$.

Note that $[f]_{W_1^1}$ is a function F from λ to ω_2:

$$F([h']_{W_1^1}) = [\beta \mapsto f(h'(\alpha),\beta)]_{W_1^1}$$

for $[h'] < [h] = \lambda$.

Fact

Every $F\colon \lambda \to \omega_2$ is represented as $F = [f]_{W^1_1}$ for some f of type h.

Fix h' with $[h'] > \sup_{\alpha,\lambda} F(\alpha)$, and let $|T_x \restriction \alpha| > \max\{h(\alpha), h'(\alpha)\}$.

For $\gamma < \omega_1$, let $\alpha_\gamma = [\beta \mapsto T_x \restriction \beta(\gamma)]$ if this is less than $h(\beta)$. Let $\beta_\gamma = F(\alpha_\gamma)$.

Let $g(\gamma) < \omega_1$ be such that $[\beta \mapsto |T_x \restriction \beta(g(\gamma))|]_{W^1_1} = \beta_\gamma$.

Then $F = [\beta \mapsto \{(|T_x \restriction \beta(\gamma), |T_x \restriction \beta(g(\gamma))|)\colon \gamma < \beta\}$.

Let $\mathcal{P}'$ partition the functions f of type h according to whether $\mathcal{P}(F) = 1$, where $F = [f]_{W_1^1}$.

Let $C \subseteq \omega_1$ be homogeneous for $\mathcal{P}'$.

Let $D = j_{W_1^1}(C) \subseteq \omega_2$. If $F \colon \lambda \to D$ is of the correct type, then there is an $f \colon \omega_1 \to C$ of type h with $F = [f]$.

This shows D is homogeneous for $\mathcal{P}$.

Non-partition results at ω_2

Theorem (Chan, J, Trang)

Let $A \subseteq \omega_2$ and suppose there is a c.u.b. $C \subseteq \omega_2$ such that $A \cap C = \mathrm{cof}_\omega \cap C$. Then $A \notin \mathrm{Ult}_{W_1^1}$.

We use the following lemma.

Lemma (almost everywhere club uniformization)

Let $f\colon \omega_1 \to \mathcal{P}(\omega_1)$ with $\forall^* \alpha\ f(\alpha)$ contains a club. Then there is a club $C \subseteq \omega_1$ such that $\forall^* \alpha \in C \setminus \{\alpha + 1\} \subseteq f(\alpha)$.

Proof: Partition $f\colon \omega_1 \to \omega_1$ of the correct type according to whether $\mathrm{ran}(f) \setminus \{f(0)\} \subseteq A_{f(0)}$. On the homogeneous side this must hold, say by C. Fix $f\colon \omega_1 \to C$ of the correct type. Then $\mathrm{ran}(f)$ witnesses the Lemma.

Lemma

Assume $\kappa \to \kappa^\kappa$. Let μ be a normal measure on κ. Let $\delta = j_\mu(\kappa)$. Then if $D \subseteq \delta$ is c.u.b., there exists a c.u.b. $C \subseteq \kappa$ with $j_\mu(C) \subseteq D$.

Proof: Partition $f, g \colon \kappa \to \kappa$ of the correct type with $f(\alpha) < g(\alpha) < f(\alpha+1)$ according to whether $[g]_\mu > N_D([f]_\mu)$.

On the homogeneous side this holds. Say $C \subseteq \kappa$ is homogeneous for this side.

Then $j_\mu(C') \subseteq D$.

Proof of Theorem: Let $C \subseteq \omega_2$ be as in the Theorem, so $A \cap \mathrm{cof}_\omega = C \cap \mathrm{cof}_\omega$. Suppose $A = [F]_{W^1_1}$, where $F(\alpha) \subseteq \omega_1$.

Let $C_0 \subseteq \omega_1$ be such that $j_{W^1_1}(C_0) \subseteq C$.

Case 1. $\forall^* \alpha\ F(\alpha)$ contains a club.

By the Lemma, let $C_1 \subseteq \omega_1$ be such that $\forall^* \alpha\ C_1 \setminus \{\alpha + 1\} \subseteq F(\alpha)$.

Let $C_2 = C_0 \cap C_1$.

Fix $f \colon \omega_1 \to C_2$ such that $f(\alpha)$ has uniform cofinality α.

Then $[f]$ has cofinality ω_1 and is in $j(C_0) \subseteq C$.

So by the assumed property of A, $[f] \notin A$.

On the other hand, $\forall^* \alpha\ f(\alpha) \in C_1 \setminus \{\alpha + 1\} \subseteq F(\alpha)$. So, $[f] \in [F] = A$.

Case 2: $\forall^* \alpha\ F(\alpha)$ is disjoint from a club.

The argument is similar, but now taking $f \colon \omega_1 \to C_2$ such that $f(\alpha)$ has uniform cofinality ω.