Club Stationary Reflection (and Friends)

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Summary and Outline

We're interested in the following general question:

General Question

How can we create models of ZFC which are simultaneously L-like and not L-like?

1. Background to the question
2. The main characters: combinatorial principles CSR, TP, SATP, AP; and some techniques (Mitchell forcing, club adding, weakly compact Laver diamonds)
3. Interactions between the characters and theorems

Goals (modest?)

The goal is that by the end of the talk you will...

1. see how this topic connects to "classic" questions in set theory
2. have a general sense of some of the techniques
3. know some open questions (if you want to work on them)

Big Picture, Broad Brush

Given a (transitive) model M of ZFC,

($*$) the "smallest" it can be is L (or a level thereof).

We'd like a variety of ways of measuring how "close" M is to L.
Lots of (related!) ways of doing this:

- Models of forcing axioms; these are "wide". You've done all forcings of a certain type along the way...
- Models of large cardinals
- comparing specific combinatorial principles between M and L.

Interested in two classes of such combinatorial principles:

Reflection and Compactness Principles

1. Reflection Principles go big-to-small: a "nice" property of a large structure also holds for many smaller substructures.
2. Compactness Principles go small-to-big: a large structure with many smaller substructures having a "nice" property also has this property.

Example

Stationary Reflection for (1); Tree Property for (2).

Not in L

L *fails* to satisfy many reflection and compactness principles. The culprit: $\square_\kappa$.

Hence, models which do satisfy them are (in that respect) non-L-like.

Refining the question:

- what kinds/amount of reflection or compactness hold in your model?
- how can we make models which realize a lot of *tension* in the sense of satisfying some compactness/reflection and failing other compactness/reflection?
- Such a model would be L-like and non-L-like simultaneously (have cake and eat it too...)

Let's move to defining the principles of interest:

1st Main Character, Stationary Reflection

Definition

Suppose $\mathrm{cf}(\beta) > \omega$. $S \subseteq \beta$ is *stationary* if $S \cap C \neq \emptyset$ for all club $C \subseteq \beta$.

(∗) stationary sets are (great for night life and) positive measure with respect to the club filter. Continuous constructions on β land in S.

Definition

A stationary $S \subseteq \beta$ *reflects* if there is $\alpha \in \beta \cap \mathrm{cof}(> \omega)$ so that $S \cap \alpha$ is stationary in α.

(∗) Positive measure set having a positive measure initial segment.

Let's get to know each other

The 1st cardinal on which stationary reflection is non-trivial is ω_2:

- No stationary subsets of ω...
- We do have stationary subsets of ω_1, but no reflection...

Another restriction we need to place on S for reflection to make sense:

Reflection Point has (relatively) High Cofinality

If all $\alpha \in S \subseteq \omega_2$ have cofinality ω_1, then S reflects nowhere:

- ♣ given a limit $\beta < \omega_2$, avoid $S \cap \beta$ by a club of countable cofinality ordinals.

No Soup (in L) For You

$\square_{\omega_1}$ implies that every stationary $S \subseteq \omega_2 \cap \text{cof}(\omega)$ contains a stationary S_0 which reflects nowhere. (Much more general than this.) Hence almost no stationary reflection in L.

Creating a Combinatorial Principle

Slogan: all (appropriate) positive measure sets having positive measure initial segments:

Definition

κ^+ satisfies the *Stationary Reflection Principle* if every stationary $S \subseteq \kappa^+ \cap \text{cof}(< \kappa)$ reflects. Denoted $\text{SR}(\kappa^+)$.

We want more! (we're doing set theory, after all) We want each appropriate S to reflect *almost everywhere*:

Definition

Say κ regular. κ^+ satisfies the *Club Stationary Reflection Principle* if for every stationary $S \subseteq \kappa^+ \cap \text{cof}(< \kappa)$, there is a club $C \subseteq \kappa^+$ so that S reflects at every $\alpha \in C \cap \text{cof}(\kappa)$. Denoted $\text{CSR}(\kappa^+)$.

Some History

1. Baumgartner (1976) showed that $\mathrm{SR}(\omega_2)$ is consistent, using a weakly compact.
    - Levy collapse a weakly compact κ to be ω_2;
    - a really nice preservation theorem shows this is sufficient.
2. Magidor (1982) showed that $\mathrm{CSR}(\omega_2)$ is consistent, from a weakly compact (optimal).
    - Levy collapse such a κ, then iterate to add clubs.
3. Harrington and Shelah (1985) showed that $\mathrm{SR}(\omega_2)$ is consistent from a Mahlo cardinal (optimal).
    - Levy collapse a Mahlo κ, then iterate to destroy "bad" stationary sets.

Introducing the Next Character: Trees

A compactness principle now (recall: small-to-big), and its negation:

Definition

A κ^+-*tree* is a tree of height κ^+ with width $\leq \kappa$.
A *cofinal branch* through a κ^+-tree is a linearly ordered subset which hits every level.
An *Aronszajn Tree* is a κ^+-tree without a cofinal branch.

Incompactness!

A κ^+-Aronszajn tree is an *in*compact object:

- it has branches of every length below κ^+, but no branch of length κ^+.
- "abrupt stop"

A Compactness Principle

We obtain a compactness principle at κ^+ by asserting that no κ^+-Aronszajn trees exist:

Definition

κ^+ satisfies the *Tree Property* if every κ^+-tree has a cofinal branch. Denoted $\mathrm{TP}(\kappa^+)$.

1. $\mathrm{TP}(\omega)$ holds (König);
2. $\mathrm{TP}(\omega_1)$ fails (Aronszajn);
3. $\mathrm{TP}(\omega_2)$ is independent: it fails under CH, and Mitchell showed its consistency (1972).

A Strong Witness to Aronszajn

Definition

A *Specializing Function* for a κ^+-tree T is a function $f : T \longrightarrow \kappa$ which is injective on chains. T is *Special* if it has a specializing function.

(∗) A specializing function for a κ^+-tree T decomposes T into a small number (i.e., κ) of simple parts (i.e., antichains).

Stubbornly Aronszajn

If T is a special κ^+ tree, then T is Aronszajn, and it remains so in any extension preserving κ^+.

An Incompactness Principle

We get a combinatorial principle by having plenty of these (note the non-triviality condition in the following):

Definition

κ^+ satisfies the *Special Aronszajn Tree Property*, denoted SATP(κ^+), if

(a) a κ^+-Aronszajn tree exists and

(b) every κ^+-Aronszajn tree is special.

1. MA implies SATP(ω_1) (Baumgartner, Malitz, Reinhardt, 1970)
2. SATP(ω_2) is consistent from a weakly compact (optimal; Laver and Shelah, 1981)
3. SATP(κ^+) for all κ regular consistent (Golshani, Hayut, 2020).

The Last Character: Approachability

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Next: a weakening of $\square_\kappa$, which is strong enough to do work (ex: stat set preservation) in ZFC.

Definition

Suppose κ is regular and $\vec{a} = \langle a_\alpha : \alpha < \kappa^+ \rangle$ is a sequence of subsets of κ^+ of size $< \kappa$. $\gamma < \kappa^+$ is *approachable w.r.t. $\vec{a}$* if

(♠) there is an unbounded $A \subseteq \gamma$ of minimal ordertype all of whose proper initial segments are enumerated before stage γ.

Definition

κ^+ satisfies the *approachability property* if there exist an $\vec{a}$ so that almost all $\gamma < \kappa^+$ are approachable w.r.t. $\vec{a}$. Denoted $\mathrm{AP}(\kappa^+)$.

- Incompactness: can't approach κ^+ in this way.

The Characters Meet

The Characters Meet

In 2018, Cummings, Friedman, Magidor, Rinot, and Sinapova published a paper "The Eightfold Way" in which they show that these are mutually orthogonal.

Note

$\square_\mu$ implies $\neg$TP(μ^+), $\neg$SR(μ^+), and AP(μ^+).

- They consider these on κ^{++} for κ either regular or countable cofinality singular.
- They show that all 8 of the Boolean combinations are consistent.

I'm interested in this flavor of problems, but obtaining models in which SR(κ^{++}) is strengthened to CSR(κ^{++}).

CSR Plays Nicely with Others

Theorem (Ben-Neria, G.; G.)

$\mathrm{TP}(\aleph_2) + \mathrm{CSR}(\aleph_2) \pm \mathrm{AP}(\aleph_2)$ *are consistent.*

Fairly straightforward, but provides a template. Other stuff with G., Levine, Stejskalova: Suslin Trees, continuum function, etc.

Theorem (G., Stejskalova)

$\mathrm{TP}(\aleph_{\omega+2}) + \mathrm{CSR}(\aleph_{\omega+2}) \pm \mathrm{AP}(\aleph_{\omega+2})$ *are consistent.*

The third has a very different flavor:

Theorem (Ben-Neria, G.)

$\mathrm{SATP}(\aleph_2) + \mathrm{CSR}(\aleph_2)$ *is consistent.*

Some Techniques

- The first two theorems use Mitchell forcing (with anticipation for the $\neg$AP cases).
- The second adds Prikry forcing and a new preservation theorem to the mix.
- The third result uses the machinery of exact strong residue functions (Neeman) to specialize trees after having added clubs.

Let's take a brief look at some varieties of Mitchell forcing. But first...

Mitchell Forcing

Mitchell developed a wonderful poset to prove that $\mathsf{TP}(\omega_2)$ is consistent. A condition has two parts: one adds Cohen reals, and the other is responsible for collapsing cardinals between ω_1 and a weakly compact κ.

Later, Uri Abraham expanded the technique to obtain $\mathsf{TP}(\omega_2) + \mathsf{TP}(\omega_3)$ by adding a third component:

- *This ensures the preservation of* $\mathsf{TP}(\omega_2)$ by further reasonable forcing. (The Universe does not like to be surprised.)

Let's take a look at the rough definition:

Mitchell Forcing with Anticipation (v1)

Say κ regular and $\lambda > \kappa$ weakly compact. Let $\mathfrak{l} : \lambda \to V_\lambda$ be arbitrary (this is a "guessing function"; more in two slides). Conditions in $\mathbb{M}_{\mathfrak{l}}(\kappa, \lambda)$ are triples (p, q, r) where

- $p \in \text{Add}(\kappa, \lambda)$ (Cohen forcing);
- $|\text{dom}(q)| \leq \kappa$ consists of inaccessibles in (κ, λ), and for all $\alpha \in \text{dom}(q)$, $q(\alpha)$ works to add a Cohen subset of κ^+ in the $\mathbb{M}_{\mathfrak{l}}(\kappa, \alpha)$-extension;
- $|\text{dom}(r)| \leq \kappa$ consists of inaccessibles in (κ, λ). IF $\gamma \in \text{dom}(r)$ and $\mathfrak{l}(\gamma)$ names a κ^+-closed poset in the $\mathbb{M}_{\mathfrak{l}}(\kappa, \gamma)$-extension, then $r(\gamma)$ is an element of $\mathfrak{l}(\gamma)$. o/w trivial.

Note

$\mathbb{M}_{\mathfrak{l}}(\kappa, \lambda)$ forces $\text{TP}(\lambda)$, $\text{AP}(\lambda)$, and $\lambda = 2^\kappa = \kappa^{++}$. The "$\mathfrak{l}$" is not necessary here.

Mitchell Forcing with Anticipation (v2)

The previous version forces AP(λ). The witnesses to AP(λ) are added by the q part which collapses frequently.

($*$) To get $\neg$AP(λ), we need to collapse sparsely:

Definition

$\mathbb{M}^*_{\mathfrak{l}}(\kappa, \lambda)$ has the same definition as $\mathbb{M}_{\mathfrak{l}}(\kappa, \lambda)$, but with the domain of q restricted to inaccessibles in (κ, λ) which are not limits of inaccessibles.

Note

$\mathbb{M}^*_{\mathfrak{l}}(\kappa, \lambda)$ forces TP(λ), $\neg$AP(λ), and $\lambda = 2^\kappa = \kappa^{++}$. This needs the "$\mathfrak{l}$" to make further forcing give CSR.

What the $\mathfrak{l}$?

λ is assumed to be weakly compact. We use the "Poor Man's Measurable" definition.

Definition (sketchy)

For "many" transitive M of size λ, there exists N transitive of size λ and an elementary $k : M \to N$ with $\text{crit}(k) = \lambda$.

Hamkins showed (using Woodin's $\mathbb{F}_\lambda$) that if λ is weakly compact, then in some extension λ also has a nice guessing property.

Definition

$\mathfrak{l} : \lambda \to V_\lambda$ is a *weakly compact Laver diamond* if for any such M, and any $A \in H(\lambda^+)$, there exists $k : M \to N$ with $k(\mathfrak{l})(\lambda) = A$.

Let's Sketch Some Proofs

Goal: give a general flavor for the ideas.

Assumptions (more later)

Work over a model of GCH in which λ is weakly compact and has a weakly compact Laver diamond $\mathfrak{l}$.

We'll do two rounds:

1. Round 1: $\mathsf{TP}(\aleph_2) + \mathsf{CSR}(\aleph_2) \pm \mathsf{AP}(\aleph_2)$;
2. Round 2: $\mathsf{TP}(\kappa^{++}) + \mathsf{CSR}(\kappa^{++}) \pm \mathsf{AP}(\kappa^{++})$ where κ is singular of cofinality ω.

Round 1

Outline, Round 1

To build our model (TP, CSR, not AP), we force with

$$\mathbb{M}^*_{\mathfrak{l}}(\omega, \lambda) * \dot{\mathbb{C}}_{\lambda^+},$$

where $\dot{\mathbb{C}}_{\lambda^+}$ is a name for a λ^+-length iteration adding clubs which witness CSR (like Magidor did). Main things to show:

1. $\mathbb{C}_{\lambda^+}$ preserves the tree property and is λ-distributive.
2. Hence forces CSR and also (by features of $\mathbb{M}^*_{\mathfrak{l}}$) forces that AP fails.

We're going to focus on a central step common to almost all stages of the argument.

The Key Maneuver

By the λ^+-chain condition, we can just look at $\mathbb{M}^*_{\mathfrak{l}} * \dot{\mathbb{C}}_\alpha$ for each $\alpha < \lambda^+$. Now we have an element of $H(\lambda^+)$.

1. we place $\mathbb{M}^*_{\mathfrak{l}} * \dot{\mathbb{C}}_\alpha$ inside some transitive M of size λ;
2. apply weak compactness to generate an embedding $k : M \to N$ with $\text{crit}(k) = \lambda$ and $k(\mathfrak{l})(\lambda) = \dot{\mathbb{C}}_\alpha$;
3. factor the forcing $k(\mathbb{M}^*_{\mathfrak{l}} * \dot{\mathbb{C}}_\alpha)$.

A key point is that by using $\mathfrak{l}$, we get:

$$k(\mathbb{M}^*_{\mathfrak{l}} * \dot{\mathbb{C}}_\alpha) \cong \mathbb{M}^*_{\mathfrak{l}} * \dot{\mathbb{C}}_\alpha * \dot{\mathbb{M}}^{\text{tail}} * k(\dot{\mathbb{C}}_\alpha).$$

$(*)$ This is necessary in order to extend the domain of k to the generic extension of M and in turn, complete the arguments.

Preservation properties of the tail forcing

Preservation properties of $\mathbb{M}^{\mathsf{tail}}$ finish the argument. $\mathbb{M}^{\mathsf{tail}}$...

1. won't add branches to λ-trees; this gives TP.
2. preserves stationary subsets of $\lambda \cap \mathrm{cof}(\omega)$; this gives distributivity of $\mathbb{C}_\alpha$, and (eventually) CSR.
3. won't collapse λ until after $2^\omega > \lambda$; this ensures (modulo the actual argument...) that AP fails.

A Template

This fairly straightforward argument forms a template for much more involved ones.

Round 2

Larger Large Cardinals

Next up: near a singular cardinal of cofinality ω.

(∗) Now a large cardinal κ will play the role of ω from Round 1. We will singularize κ using "vanilla" Prikry forcing.

We need to singularize κ after $\mathbb{M}^*_{\mathfrak{l}}(\kappa, \lambda)$, so we need that this Mitchell forcing preserves the measurability of κ.

Large Cardinal Assumption

Take κ to be indestructibly supercompact and $\lambda > \kappa$ weakly compact.

Why that order?

Let $\dot{U}$ be a $\mathbb{M}^*_{\mathfrak{l}}(\kappa, \lambda)$-name for a normal measure on κ, and we will force with:

$$\mathbb{M}^*_{\mathfrak{l}}(\kappa, \lambda) * (\text{Prikry}(\dot{U}) \times \dot{\mathbb{C}}_{\lambda^+}).$$

- This forcing is the same as doing the Prikry forcing after the club-adding (by the λ-distributivity).
- But we don't want to do $\dot{\mathbb{C}}_{\lambda^+}$ as computed in $\mathbb{M}^*_{\mathfrak{l}} * \text{Prikry}(\dot{U})$.

What does Prikry do to CSR?

We need to know that the Prikry forcing preserves the principle CSR, since the iteration $\dot{\mathbb{C}}_{\lambda^+}$ doesn't consider stationary sets added by the Prikry forcing.

A Preservation Theorem for CSR

There are plenty of other details, but we'll focus on just this one bit:

Theorem (G)

If $\mathbb{P}$ is κ^+-c.c. and κ^+-linked, then $\mathbb{P}$ preserves CSR(κ^{++})*.*
In particular, the result holds if $\mathbb{P}$ is κ-linked (ex: Vanilla Prikry with collapses...).

Recall that $\mathbb{P}$ is μ-**linked** if there is a partition $\langle \mathbb{P}^\gamma : \gamma < \mu \rangle$ of $\mathbb{P}$ so that for each $\gamma < \mu$, any two elements of $\mathbb{P}^\gamma$ are compatible in $\mathbb{P}$.

- Note that there are μ-cells, not $< \mu$-cells.

Proof of the Preservation Theorem

Let $\dot{S}$ be a $\mathbb{P}$-name for a stationary subset of $\kappa^{++} \cap \mathrm{cof}(\leq \kappa)$, and let $\langle \mathbb{P}^{\nu} : \nu < \kappa^{+} \rangle$ witness that $\mathbb{P}$ is κ^{+}-linked.
Let $\varphi : \mathbb{P} \longrightarrow \kappa^{+}$ be the function so that $p \in \mathbb{P}^{\varphi(p)}$ for all $p \in \mathbb{P}$.

For each $\nu < \kappa^{+}$, let ("trace of stems" idea)

$$T_{\nu} := \left\{ \alpha < \kappa^{++} : (\exists p \in \mathbb{P}^{\nu}) \ \left[p \Vdash \alpha \in \dot{S} \right] \right\}.$$

We say ν is **strong** if T_{ν} is stationary; note that $T_{\nu} \subseteq \kappa^{++} \cap \mathrm{cof}(\leq \kappa)$.

Proof of the Preservation Theorem

Claim: for any $p \in \mathbb{P}$, there is $q \leq p$ with $\varphi(q)$ strong.

Proof.

Fix p, and let G be V-generic containing p. Observe that

$$\dot{S}[G] \subseteq \bigcup \left\{ T_{\varphi(r)} : r \leq p \wedge r \in G \right\}.$$

But on the RHS, we have at most κ^+-many sets, and $\dot{S}[G]$ is stationary. So there is some $q \leq p$ with $q \in G$ so that $T_{\varphi(q)}$ is stationary in $V[G]$. But $T_{\varphi(q)}$ is in V, so it's stationary in V. □

For each $\nu < \kappa^+$ so that ν is strong, let $C_\nu \subseteq \kappa^{++}$ be a club so that for all $\delta \in C_\nu \cap \text{cof}(\kappa^+)$, T_ν reflects at δ.
Set

$$C := \bigcap \left\{ C_\nu : \nu \text{ is strong} \right\}.$$

Proof of the Preservation Theorem

Now we have our ingredients. We claim that $\mathbb{P}$ forces that $\dot{S}$ reflects at every $\delta \in C \cap \mathrm{cof}(\kappa^+)$.
Supposing otherwise, we may find (i) a condition $p \in \mathbb{P}$, (ii) a specific ordinal $\delta \in C \cap \mathrm{cof}(\kappa^+)$, and (iii) a club subset D of δ in V so that

$$p \Vdash \delta \cap \check{D} \cap \dot{S} = \emptyset.$$

Let $q \leq p$ so that $\varphi(q)$ is strong. Then $\delta \in C \subseteq C_{\varphi(q)}$, and so $T_{\varphi(q)}$ reflects at δ. Thus pick $\alpha \in D \cap T_{\varphi(q)} \cap \delta$. Then there is some $r \in \mathbb{P}^{\varphi(q)}$ so that $r \Vdash \alpha \in \dot{S}$. But $q, r \in \mathbb{P}^{\varphi(q)}$. So they are compatible in $\mathbb{P}$.
Let $u \leq q, r$. Then $u \Vdash \alpha \in \delta \cap \check{D} \cap \dot{S}$, which contradicts the fact that p forces that this intersection is empty.

Where do we go from here?

- Double successors of singulars of countable cofinality, from optimal LCs (need short EBFs; I can do these with "objects");
- Double successors of singulars of uncountable cofinality (need Radin forcing);
- Multiple cardinals, $\aleph_2$ and $\aleph_3$ as a test case (note: can't have CSR on two successive cardinals simultaneously!)
- Is CSR($\aleph_2$) consistent with "club isomorphisms of Aronszajn trees" on $\aleph_2$?

Thanks for listening!