Combinatorial Sets of Reals, III

Spectra and Definability

Vera Fischer

University of Vienna

May 29–June 3, 2023

Young Set Theory Workshop 2023

Definition

We refer to a MCG $\mathscr{G}$ of cardinality μ as witnesses to

$$\mu \in \mathrm{sp}(\mathfrak{a}_g) = \{|\mathscr{G}| : \mathscr{G} \text{ is mcg}\}$$

and to values $\mu \in \mathrm{sp}(\mathfrak{a}_g)$ such that

$$\aleph_1 < \mu < \mathfrak{c}$$

as intermediate cardinalities (or values).

Definition: Good projective witnesses

A good projective witness to

$$\mu \in \mathrm{sp}(\mathfrak{a}_g)$$

is a MCG $\mathscr{G}$ of cardinality μ which is also of

lowest projective complexity,

i.e. there are no witnesses to μ whose definitional complexity lies strictly below that of $\mathscr{G}$ in terms of the projective hierarchy.

Question

What can we say about the definability properties of maximal cofinitary groups $\mathscr{G}$ such that

$$\aleph_1 < |\mathscr{G}| < \mathfrak{c}?$$

Observation

Note that a Σ^1_2 MCG must be either of size $\aleph_1$ or continuum (being the union of $\aleph_1$ many Borel sets). Therefore the lowest possible projective complexity of a witness to intermediate values in $\mathrm{sp}(\mathfrak{a}_g)$ is Π^1_2.

Theorem (F., Friedman, Schrittesser, Törnquist)

It is relatively consistent with ZFC that:

- $\mathfrak{c} \geq \aleph_3$ and
- there is a Π^1_2 MCG of size $\aleph_2$.

Thus, it is consistent that there is a Π^1_2 good projective witness to an intermediate value in $\mathrm{sp}(\mathfrak{a}_g)$.

Remark

The same holds for the spectrum of MED and MAD.

Theorem (F., Friedman, Schrittesser, Törnquist)

Let $2 \leq M < N < \aleph_0$ be given. There is a cardinal preserving generic extension of the constructible universe L in which

$$\mathfrak{a}_g = \mathfrak{b} = \mathfrak{d} = \aleph_M < \mathfrak{c} = \aleph_N$$

and there is a Π^1_2 definable maximal cofinitary group fo size $\aleph_M$.

Remark

The analogous result holds for maximal families of eventually different reals, maximal families of eventually different permutations, maximal families of almost disjoint sets.

	$\aleph_1$	μ	$\mathfrak{c}$
	Π^1_1	Π^1_2	Borel
MED	✓	?	✓
MED	?	✓	✓
MCG	✓	?	✓
MCG	?	✓	✓

Independent Families

A family $\mathscr{A} \subseteq [\omega]^\omega$ is said to be independent for any two non-empty finite disjoint subfamilies $\mathscr{A}_0$ and $\mathscr{A}_1$ the set

$$|\bigcap \mathscr{A}_0 \backslash \bigcup \mathscr{A}_1| = \omega.$$

It is a maximal independent family if it is maximal under inclusion and

$$\mathfrak{i} = \min\{|\mathscr{A}| : \mathscr{A} \text{ is a m.i.f.}\}$$

- (Boolean combinations) For finite $h : \mathscr{A} \to \{0,1\}$, we refer to $\mathscr{A}^h = \bigcap h^{-1}(0) \backslash \bigcup h^{-1}(1)$ as a boolean combination. If $h' \supseteq h$, we say that $\mathscr{A}^{h'}$ strengthen $\mathscr{A}^h$.
- (Maximality) $\forall X \in ([\omega]^\omega \backslash \mathscr{A}) \exists h \in \mathrm{FF}(\mathscr{A})(X \text{ does not split } \mathscr{A})$.

... and once again Maximality

Let $\mathscr{A}$ be an independent family. If for each $X \in [\omega]^\omega \backslash \mathscr{A}$ and every $h \in \mathrm{FF}(\mathscr{A})$ there is a strengthening $\mathscr{A}^g$ of $\mathscr{A}^h$ such that X does not split $\mathscr{A}^g$, we say that $\mathscr{A}$ is densely maximal.

Remark

The notion of dense maximality appears for the first time in the work of M. Goldstern and S. Shelah on the consistency of $\mathfrak{r} < \mathfrak{u}$.

Density filter

Let $\mathscr{A}$ be an independent family. The family of all $Y \subseteq \omega$ with the property that every $\mathscr{A}^h$ has a strengthening contained in Y is a filter, referred to as the the density filter and denoted $\mathrm{fil}(\mathscr{A})$.

Lemma

Let $\mathscr{A}$ be an independent family. Then $\mathscr{A}$ is densely maximal if and only if

$$\mathscr{P}(\omega) = \mathrm{fil}(\mathscr{A}) \cup \langle \{\omega \backslash \mathscr{A}^h : h \in \mathrm{FF}(\mathscr{A})\} \rangle_{dn}.$$

Definition: Selective independence

A densely maximal independent family $\mathscr{A}$ is said to be selective if $\mathrm{fil}(\mathscr{A})$ is Ramsey.

Theorem (Shelah, 1992)

- Selective independent families exist under *CH*.
- They are indestructible by a countable support iterations and countable support products of Sacks forcing.

Remark

It is consistent that $\mathfrak{i} < \mathfrak{c}$. In fact the construction can be extracted from Shelah's proof of $\mathfrak{i} < \mathfrak{u}$.

Theorem (A. Miller)

There are no analytic maximal independent families.

Theorem (Brendle, F., Khomskii)

It is relatively consistent that $\mathfrak{i} = \aleph_1 < \mathfrak{c}$ with a co-analytic witness to $\mathfrak{i}$.

Recall that existence of a Σ^1_2 MIF implies the existence of a Π^1_1 MIF.

Optimal spectra?

	$\aleph_1$	μ	$\mathfrak{c}$	
MIF	$\checkmark$	–	?	$V^{\mathbb{S}_\lambda} \vDash \text{sp}(\mathfrak{i}) = \{\aleph_1, \mathfrak{c}\}$
MIF	–	–	$\checkmark$	$V^{\mathbb{P}} \vDash \mathfrak{r} = \mathfrak{i} = \mathfrak{c}$

It is still open how to guarantee the existence of

- good projective witnesses for two distinct cardinals in $\text{sp}(\mathfrak{i})$, or
- a good projective witness for intermediate values.

Indestructibility

Let $\mathscr{A}$ be a selective independent family. Then $\mathscr{A}$ remains selective after forcing with the countable support iteration of any of:

- (Shelah, 1989) Shelah's poset for diagonalizing a maximal ideal,
- (Cruz-Chapital, F., Guzman, Supina, 2020) Miller partition forcing,
- (J. Bergfalk, F., C. Switzer, 2021) Coding with perfect trees,
- (Switzer, 2022) *h*-perfect trees,
- (F., Switzer, 2023) Miller lite forcing,

leading in particular to the consistency of each of the following

$$\mathfrak{i} < \mathfrak{u}, \mathfrak{u} = \mathfrak{a} = \mathfrak{i} < \mathfrak{a}_T, \mathfrak{i} = \mathfrak{u} < \text{cof}(\mathscr{N}) = \text{non}(\mathscr{N}), \mathfrak{i} = \mathfrak{hm} < \mathfrak{l}_{n,\omega}.$$

Definition

A poset $\mathbb{P}$ is Cohen preserving if every every new dense open subset of $2^{<\omega}$ (or, equivalently $\omega^{<\omega}$) contains an old dense subset.

Remark

More formally, $\mathbb{P}$ is Cohen preserving if for all $p \in \mathbb{P}$ and all $\mathbb{P}$-names $\dot{D}$ so that

$$p \Vdash \text{“}\dot{D} \subseteq 2^{<\omega} \text{ is dense open”}$$

there is a dense $E \subseteq 2^{<\omega}$ in the ground model, $q \leq_{\mathbb{P}} p$ so that

$$q \Vdash \check{E} \subseteq \dot{D}.$$

Theorem (Shelah)

If δ is an ordinal and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \delta, \beta < \delta \rangle$ is a countable support iteration such that for each $\alpha < \delta$

$$\Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha \text{ is proper and Cohen preserving”}$$

then $\mathbb{P}_\delta$ is proper and Cohen preserving.

Lemma

If $\mathbb{P}$ is Cohen preserving and proper, then $\mathbb{P}$ is ${}^\omega\omega$-bounding.

Theorem (V. Fischer, C. Switzer, 2023)

Let δ be an ordinal. Let $\mathscr{A}$ be a selective independent family and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle$ be a countable support iteration of proper forcing notions so that for every $\alpha < \delta$,

$$\Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha \text{ is Cohen preserving”}.$$

If for every $\alpha < \delta$,

$$\Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha \text{ preserves the dense maximality of } \mathscr{A}\text{”}$$

then $\mathbb{P}_\delta$ preserves the selectivity of $\mathscr{A}$.

Genericity

$\mathscr{A}$-diagonalization filters

Let $\mathscr{A}$ be an independent family. A filter $\mathscr{U}$ is said to be an

$\mathscr{A}$-diagonalization filter if

$$\forall F \in \mathscr{U} \forall h \in \mathrm{FF}(\mathscr{A})(|F \cap \mathscr{A}^h| = \omega)$$

and is maximal with respect to the above property.

Genericity

Theorem (F., Montoya, Switzer 2023)

Let $\mathscr{A}$ be an independent family. Then $\mathscr{A}$ is densely maximal iff $\mathrm{fil}(\mathscr{A})$ is the unique diagonalization filter for $\mathscr{A}$.

Proof

- If $\mathscr{A}$ is densely maximal then $\mathrm{fil}(\mathscr{A})$ is

  the unique diagonalization filter for $\mathscr{A}$.

- If $\mathrm{fil}(\mathscr{A})$ is the unique diagonalization filter, then

  $\mathscr{A}$ is densely maximal.

□

Lemma

Suppose $\mathscr{U}$ is a $\mathscr{A}$-diagonalization filter, G is $\mathbb{M}(\mathscr{U})$-generic and

$$x_G = \bigcup\{s : \exists F(s,F) \in G\}.$$

Then:

1. $\mathscr{A} \cup \{x_G\}$ is independent
2. If $y \in ([\omega]^\omega \backslash \mathscr{A}) \cap V$ is such that

   $$\mathscr{A} \cup \{y\}$$

   is independent, then $\mathscr{A} \cup \{x_G, y\}$ is not independent.

Proof (1):

For $h \in \mathrm{FF}(\mathscr{A})$ and $n \in \omega$, the sets

- $D_{h,n} := \{(s, F) \in \mathbb{M}(\mathscr{U}) : |s \cap \mathscr{A}^h| > n\}$, and
- $E_{h,n} := \{(s, F) : |(\min F \setminus \max s) \cap \mathscr{A}^h| > n\}$

are dense, and so $\mathscr{A}^h \cap x_G$, and $\mathscr{A}^h \setminus x_G$ are infinite.

Proof (2):

Fix y such that $\mathscr{A} \cup \{y\}$ is independent.

1. If $y \in \mathscr{U}$, then $x_G \subseteq^* y$ and so $x_G \backslash y$ is finite.
2. If $y \notin \mathscr{U}$, then
   - either there is $F \in \mathscr{U}$ such that $F \cap y$ is finite, and so $x_G \cap y$ is finite,
   - or there are $F \in \mathscr{U}$, $h \in \mathrm{FF}(\mathscr{A})$ s.t. $F \cap y \cap \mathscr{A}^h = \emptyset$, in which case $x_G \cap y \cap \mathscr{A}^h$ is finite.
3. Thus in either case $\mathscr{A} \cup \{x_G, y\}$ is not independent.

$\square$

Corollary

Let κ be a regular uncountable cardinal. Then consistently

$$\aleph_1 < \mathfrak{i} = \kappa < \mathfrak{c}.$$

Proof:

Let $\lambda > \kappa$ be the desired size of the continuum. The ordinal product $\gamma^* = \lambda \cdot \kappa$ contains an unbounded subset $\mathscr{I}$ of cardinality κ. Consider a finite support iteration of length γ^* such that along $\mathscr{I}$ we

- recursively generate a max. independent family of cardinality κ,
- as well as a scale of length κ,

and along $\gamma^* \backslash \mathscr{I}$, we add Cohen reals. Then in the final generic extension

$$\aleph_1 < \mathfrak{d} = \kappa \leq \mathfrak{i} \leq \kappa < \mathfrak{c} = \lambda.$$

□

Genericity

Theorem (F., Switzer, 2023)

The generic maximal independent family added by an iteration of Mathias forcing relativized to diagonalization filters is selective.

Corollary

If $\mathfrak{p} = \mathfrak{c}$, then there is a selective independent family.

Genericity

Theorem (F., Switzer, 2023)

(GCH) Let $\kappa < \lambda$ be regular uncountable. It is consistent that

$$\mathfrak{i} = \kappa < \mathfrak{c} = \lambda$$

holds with a selective witness to $\mathfrak{i}$.

1. Can we adjoin via forcing a maximal ideal independent family of cardinality $\aleph_\omega$?
2. In the Sacks model $\mathfrak{sp}(\mathfrak{i}) = \{\aleph_1, \mathfrak{c}\}$.
   - Can we have a large spectrum?
   - For which sets of uncountable cardinals C can we achieve a precise evaluation $\mathfrak{sp}(\mathfrak{i}) = C$?

Lemma

Let $\mathscr{A}$ be an independent family, $\mathscr{U}$ a $\mathscr{A}$-diagonalization filter. Let $n > 1$ and for each $i \in n$ let $\mathscr{U}_i = \mathscr{U}$. Let

$$G = \prod_{i \in n} G_i \text{ be } \mathbb{P} = \prod_{i \in n} \mathbb{M}(\mathscr{U}_i)\text{-generic filter}$$

and for each $i \in n$ let $x_i = x_{G_i}$. Then in $V[G]$:

1. $\mathscr{A} \cup \{x_i\}_{i \in n}$ is independent;
2. if $y \in (V \backslash \mathscr{A}) \cap [\omega]^\omega$ be such that

$$\mathscr{A} \cup \{y\} \text{ is independent,}$$

   then for each $i \in n$, the family $\mathscr{A} \cup \{y, x_i\}$ is not independent.

Proof

Item (2) holds, since each x_i is a diagonalization real.

To prove item (1):

- fix $h \in \mathrm{FF}(\mathscr{A})$ and an arbitrary $j : n \to 2$;
- for each $n \in \omega$, we will show that the set

$$D_{h,j,n} = \{\langle (t_i, H_i)\rangle_{i\in n} : \exists i^* > n(i^* \in \bigcap t_i^{j(i)} \cap \mathscr{A}^h)\}$$

is dense in $\mathbb{P}$, where $t_i^0 = t$, $t_i^1 = \min H_i \backslash t_i$. Thus, if $p \in D_{h,j,n}$ then

$$p \Vdash i^* \in \bigcap_{i\in n} x_i^{j(j)} \cap \mathscr{A}^h,$$

where $x_i^0 = x_i$ and $x_i^1 = \omega \backslash x_i$.

Proof cnt'd:

- Let $\bar{p} = \langle (s_i, F_i) \rangle_{i \in n} \in \mathbb{P}$. Let $I = \{i \in n : j(i) = 0\}$ and $J = n \backslash I$.
- Thus, for each $i \in I$, $s_i^{j(i)} = s_i$ and for each $i \in J$, $s_i^{j(i)} = \omega \backslash s_i$.
- Since $\mathscr{U}$ is $\mathscr{A}$-diagonalization,

$$\bigcap_{i \in I} F_i \cap \mathscr{A}^h$$

is infinite and so there is

$$i^* \in \bigcap_{i \in I} F_i \cap \mathscr{A}^h,$$

which is strictly bigger than n and the maximum of s_i for all $i \in n$.

Proof cnt'd:

Then:

1. if $i \in I$, $(s_i \cup \{i^*\}, F_i \setminus (i^*+1)) \leq (s_i, F_i)$ and forces $i^* \in x_i \cap \mathscr{A}^h$;
2. if $i \in J$, $(s_i, F_i \setminus (i^*+1)) \leq (s_i, F_i)$ and forces $i^* \in (\omega \setminus x_i) \cap \mathscr{A}^h$.

Let $\bar{q} = \langle q_i \rangle_{i \in n}$ where

$$q_i = (s_i \cup \{i^*\}, F_i \setminus (i^*+1)) \text{ for } i \in I,\ q_i = (s_i, F_i \setminus (i^*+1)) \text{ for } i \in J.$$

Then $\bar{q} \leq \bar{p}$ and $\bar{q} \in D_{h,j,n}$. In particular,

$$\bar{q} \Vdash i^* \in \bigcap_{i \in n} x_i^{j(i)} \cap \mathscr{A}^h.$$

□

Proof cnt'd:

Then:

1. if $i \in I$, $(s_i \cup \{i^*\}, F_i \backslash (i^* + 1)) \leq (s_i, F_i)$ and forces $i^* \in x_i \cap \mathscr{A}^h$;
2. if $i \in J$, $(s_i, F_i \backslash (i^* + 1)) \leq (s_i, F_i)$ and forces $i^* \in (\omega \backslash x_i) \cap \mathscr{A}^h$.

Let $\bar{q} = \langle q_i \rangle_{i \in n}$ where

$$q_i = (s_i \cup \{i^*\}, F_i \backslash (i^* + 1)) \text{ for } i \in I, \; q_i = (s_i, F_i \backslash (i^* + 1)) \text{ for } i \in J.$$

Then $\bar{q} \leq \bar{p}$ and $\bar{q} \in D_{h,j,n}$. In particular,

$$\bar{q} \Vdash i^* \in \bigcap_{i \in n} x_i^{j(i)} \cap \mathscr{A}^h.$$

□

Theorem (F., Shelah)

(GCH) Let θ be an uncountable cardinal. Then, there is a ccc poset, which adjoins a maximal independent family of cardinality θ.

Remark

In particular, there is a ccc poset adjoining a maximal independent family of cardinality $\aleph_\omega$.

Proof (Outline)

Fix $\sigma \leq \theta \leq \lambda$, where:

- σ is regular uncountable (the intended value of $\mathfrak{i}$),
- λ is of uncountable cofinality (the intended value of $\mathfrak{c}$).
- Let $S \subseteq \theta^{<\sigma}$ be a well-prunded θ-splitting tree of height σ.
- For each $\alpha < \sigma$, let S_α be the α-th splitting level of S.

Recursively define a finite support iteration

$$\mathbb{P}_S = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \sigma, \beta < \sigma \rangle$$

of length σ such that for each α, in $V^{\mathbb{P}_\alpha}$ we have

$$\mathbb{Q}_\alpha = \prod_{\eta \in S_\alpha} \mathbb{Q}_\eta$$

where $\mathbb{Q}_\eta$ is Mathias forcing for an appropriate diagonalization filter. Moreover the diagonalization filters are chosen in such a way, that in $V^{\mathbb{P}_S}$ for each branch $\eta \in [S]$ the family

$$\mathscr{A}_\eta = \{a_\nu : \nu \in \text{succ}(\eta \restriction \xi), \xi < \alpha\}$$

is a maximal independent family of cardinality θ. □

Corollary (F., Shelah)

There is a ccc forcing notion adjoining a maximal independent family $\mathscr{A}$ such that

$$|\mathscr{A}| = \aleph_\omega.$$

Proof:

Use an $\aleph_\omega$-splitting tree of height ω_1. □

Theorem (F., Shelah, 2022)

Assume GCH.Let σ be a regular uncountable cardinal, λ a cardinal of uncountable cofinality such that $\sigma \leq \lambda$. Let

$$\Theta_1 \subseteq [\sigma, \lambda]$$

be such that

$$\sigma = \min \Theta_1, \max \Theta_1 = \lambda.$$

Then there is a ccc generic extension in which

$$\Theta_1 \subseteq \mathfrak{sp}(\mathfrak{i}).$$

Theorem (F., Shelah)

- For any finite set $C \subseteq \{\aleph_n\}_{n \in \omega \setminus 1}$, consistently
$$\mathrm{sp}(\mathfrak{i}) = C.$$
- For any infinite $C \subseteq \{\aleph_n\}_{n \in \omega \setminus 1}$ and $\lambda > \aleph_\omega$ of uncountable cofinality, consistently
$$\mathrm{sp}(\mathfrak{i}) = C \cup \{\aleph_\omega, \mathfrak{c} = \lambda\}.$$

Comment

Excluding values is an isomorphism of names argument, essentially a counting argument, relying on specific properties of the forcing construction.

Ideal Independence

- A family $\mathscr{A} \subseteq [\omega]^\omega$ such that for all $\mathscr{F} \in [\mathscr{A}]^{<\omega}$ and $A \in \mathscr{A} \setminus \mathscr{F}$, the set
$$A \setminus \bigcup \mathscr{F}$$
is infinite, is said to be ideal independent.
- An ideal independent family which is maximal under inclusion is said to be a maximal ideal independent family.
- The least cardinality of an infinite ideal independent family, maximal under inclusion, is denoted $\mathfrak{s}_{mm}$.
- Almost disjoint families and independent families are both examples of ideal independent families.

Earlier investigations (Cancino, Guzman, Miller) of $\mathfrak{s}_{mm}$ show that

$$\max\{\mathfrak{d}, \mathfrak{r}\} \leq \mathfrak{s}_{mm}$$

and that each of the following inequalities

$$\mathfrak{u} < \mathfrak{s}_{mm}, \quad \mathfrak{s}_{mm} < \mathfrak{i}, \quad \mathfrak{s}_{mm} < \mathfrak{c}$$

is consistent.

Definition

Let $\mathscr{A}$ be an ideal independent family. For any $A \in \mathscr{A}$, the filter $\mathscr{F}(\mathscr{A}, A)$ generated by the family

$$\{A \setminus \bigcup \mathscr{F} : \mathscr{F} \in [\mathscr{A}]^{<\omega} \wedge A \notin \mathscr{F}\}.$$

is referred to as the *complemented filter of* $\mathscr{A}$

Observation

An ideal independent family $\mathscr{A}$ is maximal if and only if

$$\mathscr{P}(\omega) = \mathscr{I}(\mathscr{A}) \cup \left(\bigcup \{ \mathscr{F}(\mathscr{A}, A) : A \in \mathscr{A} \} \right).$$

Theorem(Bardyla, Cancino, F., Switzer)

$\mathfrak{u} \leq \mathfrak{s}_{mm}$

As a consequence we obtain the independence of $\mathfrak{s}_{mm}$ and $\mathfrak{i}$, as

- the consistency of $\mathfrak{s}_{mm} < \mathfrak{i}$ is shown by Cancino, Guzman, Miller,
- while the consistency of $\mathfrak{i} < \mathfrak{s}_{mm}$ follows from the above result and the consistency of $\mathfrak{i} < \mathfrak{u}$.

Definition

Let $\mathscr{U}$ be an ultrafilter. A maximal ideal independent family $\mathscr{A}$ is called $\mathscr{U}$-*encompassing* if the following conditions hold:

1. $\mathscr{U} \cap \mathscr{A} = \emptyset$, i.e. $\mathscr{A}$ is contained in the dual ideal of $\mathscr{U}$.
2. For every $X \in \mathscr{U}$ the set of $A \in \mathscr{A}$ so that $X \in \mathscr{F}(\mathscr{A}, A)$ is co-countable.

Theorem (Bardyla, Cancino, F., Switzer)

Assume CH. For any p-point $\mathscr{U}$ there is a $\mathscr{U}$-encompassing maximal ideal independent family $\mathscr{A}$ such that for all $A \in \mathscr{A}$, the filter $\mathscr{F}(\mathscr{A}, A)$ is a p-point.

Theorem (Bardyla, Cancino, F., Switzer)

Let $\mathscr{U}$ be a p-point and let $\mathbb{P}$ be

$$\text{proper, } \omega^\omega\text{-bounding, } p\text{-points preserving poset.}$$

Then $\mathbb{P}$ preserves the maximality of any

$$\mathscr{U}\text{-encompassing maximal ideal independent family } \mathscr{A}$$

with the property that $\mathscr{F}(\mathscr{A}, A)$ is a p-point for all $A \in \mathscr{A}$.

Observation

Note that this theorem implies that under CH, in the generic extension by any proper, ω^ω-bounding, p-point preserving forcing notion $\mathfrak{s}_{mm} = \aleph_1$.

Corollary

1. $\mathfrak{s}_{mm} = \aleph_1$ in the Sacks model.
2. $\mathfrak{s}_{mm} = \aleph_1$ in the Miller partition model and hence $\mathfrak{s}_{mm} < \mathfrak{a}_T$ is consistent.
3. $\mathfrak{s}_{mm} = \aleph_1$ in the *h*-perfect tree forcing model and hence $\mathfrak{s}_{mm} < \mathrm{non}(\mathcal{N})$ is consistent.

An alternation of Miller partition forcing and *h*-perfect tree forcings leads to the consistency of

$$\mathfrak{i} = \mathfrak{s}_{mm} < \text{non}(\mathcal{N}) = \mathfrak{a}_T = \aleph_2.$$

Corollary

$\mathfrak{s}_{mm}$ is independent of $\mathfrak{a}_T$

Proof.

- In the Miller partition model, $\mathfrak{s}_{mm} < \mathfrak{a}_T$.
- On the other hand, $\mathfrak{a}_T < \mathfrak{u}$ holds in the Random model and hence $\mathfrak{a}_T < \mathfrak{s}_{mm}$ holds in that model as well.

□

Lemma: Eliminating intruders

Let $\mathscr{A}$ be an ideal independent family. There is a *ccc* forcing $\mathbb{P}(\mathscr{A})$ which adds a set z such that in $V^{\mathbb{P}(\mathscr{A})}$:

1. $\mathscr{A} \cup \{z\}$ is an ideal independent family, and
2. for each $y \in V \cap ([\omega]^\omega \setminus \mathscr{A})$ the family $\mathscr{A} \cup \{z, y\}$ is not ideal independent.

Theorem

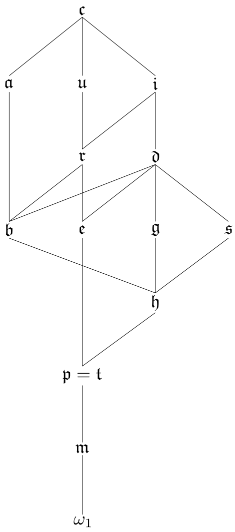
Assume *GCH*. Let C be a set of uncountable cardinals. Then there is a *ccc* generic extension in which

$$C \subseteq \mathrm{spec}(\mathfrak{s}_{mm}) = \{|\mathscr{A}| : \mathscr{A} \text{ is a maximal ideal independent family}\}.$$

Under some restrictions on C, one can obtain a cardinal preserving generic extension in which C is realized as $\mathrm{spec}(\mathfrak{s}_{mm})$.

Questions

- Is it consistent that $\mathfrak{s}_{mm} = \aleph_\omega$?
- What *ZFC* restrictions are there on the set $\text{spec}(\mathfrak{s}_{mm})$?


$\mathfrak{c}$
$\mathfrak{a}$
$\mathfrak{u}$
$\mathfrak{i}$
$\mathfrak{r}$
$\mathfrak{d}$
$\mathfrak{b}$
$\mathfrak{e}$
$\mathfrak{g}$
$\mathfrak{s}$
$\mathfrak{h}$
$\mathfrak{p} = \mathfrak{t}$
$\mathfrak{m}$
ω_1

Thank you for your attention!