Combinatorial Sets of Reals, I

Creatures, Templates and Coherent Systems

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Classical real analysis & combinatorial sets of reals:

The works of

- Du Bois-Reymond, Riemann, Cantor,
- Hausdorff, Hilbert, Rothberger,
- Sierpinski, Souslin, Borel,
- many others, as well as
- many of our contemporaries

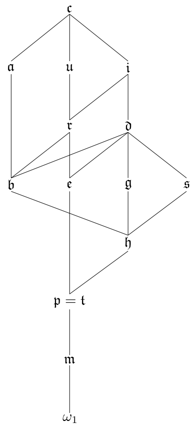
leading to interesting combinatorial structures on $\mathbb{N}^{\mathbb{N}}$ and $[\mathbb{N}]^{\infty}$.

Elementary(?!) combinatorics of $[\mathbb{N}]^\infty$ and $\mathbb{N}^\mathbb{N}$:

Combinatorics of eventual dominance, $\leq^*$, and basic set theoretic operations on the above spaces:

- unbounded families
- dominating families
- almost disjointness
- independence
- towers
- refining systems of a.d. families
- splitting families
- ideal independent families
- $\ldots$

Some classical characteristics



Independence

The infinitary combinatorics captured by the above sets of reals is complex and certainly testing the boundaries of ZFC, as witnessed by the abundance of independence among these structures, and the associated combinatorial cardinal characteristics.

A richness of combinatorial structures and four directions of research

- ZFC dependencies and independence; Constellations;
- Spectra and Strong Maximality
- Projective complexity
- Combinatorics at the uncountable and the so called higher Baire spaces

$\mathfrak{i}$ vs. $\mathfrak{u}$

In the Miller model $\mathfrak{u} < \mathfrak{i}$, while Shelah devised a special ${}^\omega\omega$-bounding poset the countable support iteration of which produces a model of $\mathfrak{i} = \aleph_1 < \mathfrak{u} = \aleph_2$ (and an instance of strong maximality).

$\mathfrak{a}$ vs. $\mathfrak{u}$

In the Cohen model $\mathfrak{a} < \mathfrak{u}$, while assuming the existence of a measurable one can show the consistency of $\mathfrak{u} < \mathfrak{a}$. The use of a measurable has been eliminated by Guzman and Kalajdzievski.

$\mathfrak{a}$ vs $\mathfrak{i}$

In the Cohen model $\mathfrak{a} < \mathfrak{i} = \mathfrak{c}$.

Question:

Is it consistent that $\mathfrak{i} < \mathfrak{a}$?

Theorem (J. Cruz-Chapital, V.F., O. Guzman, J, Supina, 2022)

It is relatively consistent, that $\mathfrak{i} < \mathfrak{a}_T$.

- (Kuratowski)

  Definition: A λ-set is a set $A \subseteq \mathbb{R}^n$ with the property that every countable subset is relative G_δ.

- Let $\mathscr{A}(\lambda\text{-set})$ denote the collection of all λ-sets.

- (1937 Sierpinski)

  Question: Is $\mathscr{A}(\lambda\text{-set})$ countably additive?

  Sufficient condition: $\forall A \in \mathscr{A}(\lambda\text{-set}) \forall X \in [\mathbb{R}^n]^{\leq\omega}(A \cup X \in \mathscr{A}(\lambda\text{-set}))$

  Definition: $X \in \mathscr{A}(\lambda'\text{-set})$ if $X \cup Y \in \mathscr{A}(\lambda\text{-set})$ for every $Y \in [\mathbb{R}]^{\leq\omega}$.

  Question: Is it the case that $\mathscr{A}(\lambda\text{-set}) \subseteq \mathscr{A}(\lambda'\text{-set})$?

- (1939 Rothberger) (AC) $\exists X \in \mathscr{A}(\lambda'\text{-set}) \setminus \mathscr{A}(\lambda\text{-set})$
- (Some crucial ideas from the proof)
  - Rothberger isolated the property

    $$B(\aleph_\xi),$$

    which holds in case every family of sequences of natural numbers of cardinality $\aleph_\xi$ is bounded.
  - and defined $\aleph_\eta$ to be the least cardinal for which $B(\aleph_\eta)$ fails.
- Thus, in contemporary notation $\mathfrak{b} = B(\aleph_\eta)$.

- (1974 Booth) For every regular uncountable cardinal λ

  the space 2^λ is sequentially compact

  iff for every sequence $\langle a_\alpha : \alpha < \lambda \rangle \subseteq [\mathbb{N}]^\omega$ there is a set $b \subseteq \mathbb{N}$ such that

  $$b \subseteq^* a_\alpha \text{ or } b \subseteq^* \mathbb{N} \setminus a_\alpha \text{ for all } \alpha < \lambda.$$

- Thus in contemporary notation:

  2^λ is sequentially compact iff $\lambda < \mathfrak{s}$.

The bounding and the splitting numbers

- Balcar, Pelant, Simon established the consistency of $\mathfrak{s} < \mathfrak{b}$.
- In the lack of a ZFC proof of $\mathfrak{s} \leq \mathfrak{b}$, S. Shelah introduced the powerful technique of creature forcing, establishing the consistency of
$$\mathfrak{b} = \aleph_1 < \mathfrak{s} = \aleph_2.$$

Start over a model of CH. Proceed with a countable support iteration of proper forcing notions $\langle \mathbb{P}_\alpha : \alpha \leq \omega_2 \rangle$. For each $\alpha \leq \omega_2$ let $V_\alpha = V^{\mathbb{P}_\alpha}$ and let $V = V_0$ be the ground model then:

1. $\forall \alpha < \omega_2 \exists r_{\alpha+1} \in V_{\alpha+1} \cap [\omega]^\omega$ such that

$$r_{\alpha+1} \text{ is not split by } V_\alpha \cap [\omega]^\omega \text{ and}$$

2. $V_0 \cap {}^\omega\omega$ remains unbounded in $V_{\omega_2} \cap {}^\omega\omega$.

Then no family of cardinality $\aleph_1$ is splitting, as for any such family $\mathscr{A}$ there is $\alpha < \omega_2$ such that $\mathscr{A} \subseteq V_\alpha \cap [\omega]^\omega$ and so $r_{\alpha+1}$ is not split by $\mathscr{A}$. Thus, $\mathfrak{s} = \aleph_2$, while by property (2) we have $\mathfrak{b} = \aleph_1$ in the final model.

First attempt!

Mathias forcing:

1. $(s, E) \in [\omega]^{<\omega} \times [\omega]^{\omega}$, $\max s < \min E$
2. $(s_1, E_1) \leq (s_2, E_2)$ iff s_1 end-extends s_2, $s_1 \backslash s_2 \subseteq E_2$, $E_1 \subseteq E_2$.

If G is $\mathbb{M}$ generic over V then

- $r_G = \bigcup\{s : \exists A (s, A) \in G\}$ is un-split by $V \cap [\omega]^{\omega}$,
- however its enumerating function dominates $V \cap {}^{\omega}\omega$.

Definition (Finite logarithmic measures, Shelah 1984)

Let $x \in [\omega]^{<\omega}$. A function

$$h : \mathscr{P}(x) \to \omega$$

is said to be a finite logarithmic measure on x if

$$\text{whenever } x = x_0 \cup x_1 \text{ then } h(x_0) \leq h(x) - 1 \text{ or } h(x_1) \leq h(x) - 1,$$

unless $h(x) = 0$. The value $h(x)$ is called the level of the measure.

Definition

Let $\mathbb{Q}$ be the set of all pairs (u, T) where $u \in [\omega]^{<\omega}$,

$$T = \langle (s_i, h_i) : i \in \omega \rangle$$

is a sequence of finite logarithmic measures, such that

1. $\max u < \min s_0$,
2. $\max s_i < \min s_{i+1}$,
3. $h_i(s_i) < h_{i+1}(s_{i+1})$.

Let

$$\mathrm{int}(T) = \bigcup \{s_i : i \in \omega\}.$$

Remark

Note, if $(u, T) \in \mathbb{Q}$, then $(u, \mathrm{int}(T)) \in \mathbb{M}$.

Definition (continued)

$(u_2, T_2) \leq (u_1, T_1)$ where $T_k = \langle t_i^k : i \in \omega \rangle$ for $k = 1, 2$, $t_i^k = (s_i^k, h_i^k)$ if

1. u_2 end-extends u_1 and $u_2 \setminus u_1 \subseteq \text{int}(T_1)$
2. $\text{int}(T_2) \subseteq \text{int}(T_1)$ and there is a sequence $\langle B_i : i \in \omega \rangle \subseteq [\omega]^{<\omega}$ s. t.
   - $\max u_2 < \min s_j^1$ for $j = \min B_0$, $\max B_i < \min B_{i+1}$, $s_i^2 \subseteq \bigcup \{ s_j^1 : j \in B_i \}$.
   - for every $e \subseteq s_i^2$ such that $h_i^2(e) > 0$

     there is $j \in B_i$ such that $h_j^1(e \cap s_j^1) > 0$.

Properties

1. $\mathbb{Q}$ is Axiom A.
2. Let $A \in [\omega]^\omega$. Then
$$D_A = \{(u, T) \in \mathbb{Q} : \text{int}(T) \subseteq A \text{ or } \text{int}(T) \subseteq A^c\}$$
is dense and so
$$u_G = \bigcup \{u : \exists T (u, T) \in G\}$$
is either contained in A or in A^c. Thus, u_G is not split by $V \cap [\omega]^\omega$.
3. The countable support iteration of $\mathbb{Q}$ preserves the ground model reals unbounded. In fact, the poset is almost ${}^\omega\omega$-bounding.

Definition: Almost ${}^\omega\omega$-bounding

The partial order $\mathbb{P}$ is almost ${}^\omega\omega$-bounding if

- for every $\mathbb{P}$-name $\dot{f}$ for a function in ${}^\omega\omega$ and
- every condition $p \in \mathbb{P}$

there is a ground model function $g \in {}^\omega\omega$ such that for every infinite subset of A of ω there is an extension q_A of p such that

$$q_A \Vdash \exists^\infty k \in A(\dot{f}(k) \leq \check{g}(k)).$$

Theorem (CH)

The countable support iteration of proper, almost ${}^{\omega}\omega$-bounding posets is weakly bounding.

Theorem (Shelah 1984)

It is relatively consistent that $\mathfrak{b} = \aleph_1 < \mathfrak{s} = \aleph_2$.

$\cdots$ which establishes the independence of $\mathfrak{b}$ and $\mathfrak{s}$.

It is worth pointing out that every admissible assignment of $\aleph_1$ and $\aleph_2$ to the cardinal invariants of measure and category in the Cichon diagram can be realized in a generic extension via a countable support iteration of proper posets.

$\cdots$ beyond $\mathfrak{c} = \aleph_2$

The consistency of $\mathfrak{d} = \aleph_1 < \mathfrak{a} = \aleph_2$ is one of the most difficult, persistent problems of the combinatorial cardinal characteristics of the continuum. Note that

- while $\mathfrak{a} < \mathfrak{d}$ holds in the Cohen model,
- the consistency of $\aleph_1 < \mathfrak{d} < \mathfrak{a}$ was obtained only after S. Shelah introduced the method of template iterations.

Moreover the consistency of $\mathfrak{d} = \aleph_1 < \mathfrak{a} = \aleph_2$ is still open.

Theorem (S. Shelah, 1996; published 2000)

(GCH) It is relatively consistent that $\aleph_1 < \mathfrak{d} < \mathfrak{a} = \mathfrak{c}$.

In fact, in Shelah's model of $\mathfrak{d} < \mathfrak{a}$, the following holds:

$$\mathfrak{s} = \aleph_1 < \mathfrak{d} = \mathfrak{b} < \mathfrak{a}.$$

In 2016, introducing a new dimension to Shelah's template construction, the notion of a width of a template, V.F. and D. Mejia generalized the above and in particular obtained the following:

$$\aleph_1 < \mathfrak{s} < \mathfrak{b} = \mathfrak{d} < \mathfrak{a} = \mathfrak{c}.$$

1. It seems that, by 2016, in fact a bit earlier, ccc extensions in which $\mathfrak{c} > \aleph_2$ slowly started gaining interest.
2. A question, that should be mentioned at this point is the following:

   Can $\mathfrak{a}$ be of countable cofinality?

3. Since $\mathfrak{c}$ cannot be of countable cofinality (König, Hilbert), the question clearly calls for a model of $\aleph_\omega < \mathfrak{c}$.

The question was answered to the positive, by J. Brendle, who,

- modified Shelah's template iteration construction producing $\aleph_1 < \mathfrak{d} < \mathfrak{a}$ in such a way,
- that, Hechler's poset for adding a mad family of cardinality $\aleph_\omega$, $\mathbb{H}(\aleph_\omega)$, appears a complete suborder of the modified construction.

This, led to the consistency of

$$\aleph_1 < \mathfrak{b} = \mathfrak{d} < \mathfrak{a} = \aleph_\omega < \mathfrak{c}$$

Definition (Two-sided template, Brendle)

A two-sided template is a 4-tuple

$$\mathscr{T} = ((L, \leq_L), \mathscr{I}, L_0 \cup L_1)$$

where $(L, \leq)$ is a l.o., $\mathscr{I} \subseteq \mathscr{P}(L)$ and L_0, L_1 partition L so that

- $\mathscr{I}$ is closed under finite intersections, unions, $\emptyset$, $L \in \mathscr{I}$
- If $x <_L y$, where $x \in L$ and $y \in L_1$ then

$$\exists A \in \mathscr{I} \big(x \in A \wedge A \subseteq L_y = \{z \in L : z <_L y\} \big).$$

- If $A \in \mathscr{I}$ and $\in L_1 \backslash A$, then $A \cap L_x \in \mathscr{I}$.
- $\{A \cap L_1 : A \in \mathscr{I}\}$ is well-founded by inclusion.
- All $A \in \mathscr{I}$ are L_0-closed, i.e.

$$A = \mathrm{cl}_{L_0}(A) = A \cup \bigcup_{x \in A} L_x \cap L_0.$$

Define a rank function $\mathrm{Dp} : \mathscr{I} \to \mathbb{ON}$ by letting

- $\mathrm{Dp}(A) = 0$ for $A \subseteq L_0$,
- $\mathrm{Dp}(A) = \sup\{\mathrm{Dp}(B) + 1 : B \in \mathscr{I}, B \cap L_1 \subset A \cap L_1\}$.

The rank of $\mathscr{T}$, denoted $\mathrm{Rk}(\mathscr{T})$, is defined to be $\mathrm{Dp}(L)$.

If $A \subseteq L$, then $\mathscr{T}_A = ((A, \leq), \mathscr{I} \upharpoonright A, L_0 \cap A, L_1 \cap A)$, where

$$\mathscr{I} \upharpoonright A = \{A \cap B : B \in \mathscr{I}\}.$$

Note that:

- If $A \in \mathscr{I}$ then $\mathrm{Rk}(\mathscr{T}_A) = \mathrm{Dp}(A)$.
- If $A \subseteq L$ is arbitrary, then $\mathrm{Rk}(\mathscr{T}_A) \leq \mathrm{Rk}(\mathscr{T})$.

Definition* (Iteration along a template)

Let $\mathscr{T} = ((L, \leq), \mathscr{I}, L_0, L_1)$ be a two-sided template. Let

1. $\mathbb{Q}$ be a finite function poset with the strong embedding property such that $L_0 = \mathrm{dom}(\mathbb{Q})$,
2. $\mathbb{S}$ a good σ-Suslin poset.

The poset $\mathbb{P}(\mathscr{T}, \mathbb{Q}, \mathbb{S})$ is defined recursively as follows:

1. If $\mathrm{Rk}(\mathscr{T}) = 0$, then $\mathbb{P}(\mathscr{T}, \mathbb{Q}, \mathbb{S}) = \mathbb{Q}$.
2. Let $\mathrm{Rk}(\mathscr{T}) = \kappa$ and for all $B \in \mathscr{I}(\mathrm{Rk}(B) < \kappa)$, $\mathbb{P}_B = (\mathscr{T}_B, \mathbb{Q}, \mathbb{S})$ has been defined.

Then $\mathbb{P} = \mathbb{P}(\mathscr{T}, \mathbb{Q}, \mathbb{S})$ consists of conditions $P = (p, F^p)$ where

- p is a finite partial function, $\mathrm{dom}(p) \subseteq L$,
- $(p \restriction L_0, W^p) \in \mathbb{Q}$ and
- if $x_p = \max\{\mathrm{dom}(p) \cap L_1\}$ is defined, then there is $B \in \mathscr{I}_{x_p}$ such that

  a natural restriction $P \restriction L_{x_p} \in \mathbb{P}_B$ and $p(x_p)$ is a nice $\mathbb{P}_B$-name

  such that
  $$(P \restriction L_{x_p}, p(x_p)) \in \mathbb{P}_B * \dot{\mathbb{S}}.$$

B is called a witness to $P \in \mathbb{P}$.

Define $Q \leq_{\mathbb{P}} P$ if

$$\mathrm{dom}(p) \subseteq \mathrm{dom}(q), (q \restriction L_0, F^p) \leq_{\mathbb{Q}} (p \restriction L_0, F^p)$$

and if $x_p = \max\{\mathrm{dom}(p) \cap L_1\}$ is defined, then

- either $x_p < x_q$ and $\exists B \in \mathscr{I}_{x_q}$ such that $P \restriction L_{x_q}, Q \restriction L_{x_q} \in \mathbb{P}_B$ and $Q \restriction L_{x_q} \leq_{\mathbb{P}_B} P \restriction L_{x_q}$,
- or $x_p = x_q$ and there is $B \in \mathscr{I}_{x_q}$ witnessing $P, Q \in \mathbb{P}$ such that

$$(Q \restriction L_{x_q}, q(x_q)) \leq_{\mathbb{P}_B * \dot{\mathbb{S}}} (P \restriction L_{x_p}, p(x_p)).$$

1. V.F. and A. Törnquist developed an analogue of $\mathbb{H}(\gamma)$, Hechler's poset for adding an almost disjoint family of size γ, maximal for $\gamma \geq \omega_1$, for
   - eventually different families of functions,
   - eventually different families of permutations,
   - cofinitary groups.
2. These posets can be used to obtain a modified template construction and produce the consistency of each of (V.F., A. Törnquist, 2015):

$$\mathfrak{a}_e = \aleph_\omega, \mathfrak{a}_p = \aleph_\omega, \mathfrak{a}_g = \aleph_\omega.$$

Thank you for your attention!