Chang's Conjecture for triples revisited

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Jónsson cardinals

Definition

A cardinal κ is *Jónsson* if for all structures $\mathfrak{A}$ on κ in a countable language, there is $X \prec \mathfrak{A}$ such that $|X| = \kappa$ but $X \neq \kappa$.

Facts:

1. Ramsey cardinals and singular limits of measurables are Jónsson.
2. ω is not Jónsson, because of the function $n \mapsto n - 1$.
3. If κ is not Jónsson, then neither is κ^+.
4. (Tryba) Successors of regular cardinals are not Jónsson.
5. (Shelah) If κ is singular and not the limit of regular Jónsson cardinals, then κ^+ is not Jónsson. In particular, $\aleph_{\omega+1}$ is not Jónsson.

So the least possible cardinal that could be Jónsson is $\aleph_\omega$. The question of whether this is consistent is one of the top open problems in set theory, according to Wikipedia!

How to make $\aleph_\omega$ Jónsson

Axiom I2

There is an elementary embedding $j : V \to M$ with critical point κ, such that, if $\kappa_1 = \kappa$ and $\kappa_{n+1} = j(\kappa_n)$ and $\lambda = \sup_n \kappa_n$, then $V_\lambda \subseteq M$.

Assume we have such an embedding. We want to collapse each κ_n to be ω_n and generically lift the embedding.

Assume we have such a forcing $\mathbb{P}$ that does this, so that whenever $G \subseteq \mathbb{P}$ is generic, then there is a further forcing which yields a $G' \subseteq j(\mathbb{P})$ that is generic over M, with $j``G \subseteq G'$. So we can lift to $j : V[G] \to M[G']$.

Work in $V[G]$. Let $\mathfrak{A}$ be a rich enough structure on λ. Let $F : [\lambda]^{<\omega} \to \lambda$ be such that whenever $X \subseteq \lambda$ is closed under F, then $X \prec \mathfrak{A}$, and further that F is closed under compositions, (meaning that for all $X \subseteq \lambda$, $F``[X]^{<\omega}$ is closed under F).

How to make $\aleph_\omega$ Jónsson

For each $n < \omega$, let F_n be the function on $[\kappa_n]^{<\omega}$ where $F_n(x) = F(x)$ if $F(x) < \kappa_n$ and otherwise $F_n(x) = 0$. So $X \subseteq \lambda$ is closed under F iff for each n, $X \cap \kappa_n$ is closed under F_n.

Let T be the tree of sets X such that for some n, $X \subseteq \kappa_n$, X is closed under F_n, $X \cap \kappa_1 \in \kappa_1$, and for $1 < i \leq n$, $|X \cap \kappa_i| = \kappa_{i-1}$. We put $X \leq_T Y$ when $X = Y \cap \kappa_n$, where n is the witness that $X \in T$.

In the extension where we get G', $j``\lambda$ is closed under $j(F)$. $M[G']$ does not see $j``\lambda$, but it sees that for each n, $X_n = j``\lambda \cap \kappa_n \in j(T)$, and $X_n <_{j(T)} X_{n+1}$. By absoluteness of well-foundedness, $j(T)$ has an infinite branch in $M[G']$.

By elementarity, T has an infinite branch in $V[G]$, and the union of this branch witnesses Jónssonness of $\lambda = \aleph_\omega^{V[G]}$.

OK, but how do we get such a $\mathbb{P}$?

f-Jónssonness

If $\theta > \aleph_\omega$ is regular and $M \prec H_\theta$ has $|M \cap \aleph_\omega| = \aleph_\omega$, let $\chi_M(n)$ be the number k such that $\mathrm{ot}(M \cap \omega_k) = \omega_n$.

Definition

Let $f : \omega \to \omega$. $\aleph_\omega$ is *f-Jónsoson* if for regular $\theta > \aleph_\omega$, stationary-many $M \prec H_\theta$ have $\chi_M = f$.

If something like this forcing approach works, it would get $\aleph_\omega$ is s_1-Jónsson, where $s_1(0) = 0$ and $s_1(n) = n + 1$ for $n > 0$.

Theorem (E.)

If $\aleph_\omega$ is s_1-Jónsson, then it is f-Jónsson for all increasing $f : \omega \to \omega$ with $f(0) = 0$. (Note that if $\aleph_\omega$ is f-Jónsson, then f takes this form.)

f-Jónssonness

Theorem (Silver)

If $2^\omega < \aleph_\omega$ and $\aleph_\omega$ is Jónsson, then $\aleph_\omega$ is f-Jónsson for some f.

Proof sketch: Let $\mathfrak{A} \prec \langle H_\theta, \in, \lhd \rangle$, where $\theta > \aleph_\omega$ and $\lhd$ is a well-order. Let $2^\omega = \omega_n$. If $M \prec \mathfrak{A}$, then we can take $N = \mathrm{Sk}^{\mathfrak{A}}(M \cup \omega_n)$, and we will have $\sup(N \cap \omega_k) = \sup(M \cap \omega_k)$ for all $k > n$. This is because, if g is a definable Skolem function, $p \in M$, and $\alpha \in \omega_n$, then

$$g(p, \alpha) < \sup_{\beta < \omega_n} g(p, \beta) \in M \cap \omega_k.$$

Thus there are stationary-many $M \prec \mathfrak{A}$ with $\omega^\omega \subseteq M$, $|M \cap \aleph_\omega| = \aleph_\omega$ and $M \cap \aleph_\omega \neq \aleph_\omega$. The function $M \mapsto \chi_M$ is regressive on a stationary set, and thus constant on one by Fodor. $\square$

Another way of writing that $\aleph_\omega$ is f-Jónsson is:

$$(\ldots, \aleph_{f(3)}, \aleph_{f(2)}, \aleph_{f(1)}, \aleph_{f(1)-1}) \twoheadrightarrow (\ldots, \aleph_3, \aleph_2, \aleph_1, \aleph_0)$$

If m_0 is least such that the m_0^{th} corresponding number on the left is greater than m_0, then recursively defining $m_1 =$ that number and $m_{n+1} = f(m_n)$, we have:

$$(\ldots, \aleph_{m_4}, \aleph_{m_3}, \aleph_{m_2}, \aleph_{m_1}) \twoheadrightarrow (\ldots, \aleph_{m_3}, \aleph_{m_2}, \aleph_{m_1}, \aleph_{m_0})$$

In particular, if $\aleph_\omega$ is s_1-Jónsson, then

$$(\ldots, \aleph_4, \aleph_3, \aleph_2, \aleph_1) \twoheadrightarrow (\ldots, \aleph_3, \aleph_2, \aleph_1, \aleph_0)$$

Since these "Chang principles" are transitive, this implies $(\aleph_{n+k}, \ldots, \aleph_m) \twoheadrightarrow (\aleph_{m+k}, \ldots, \aleph_m)$ for all $n, m, k < \omega$ with $n > m$. Let's see how much of this we can get.

Theorem (Foreman, 1983)

It is consistent relative to a 2-huge cardinal that for all $m < n < \omega$, $(\aleph_{n+1}, \aleph_n) \twoheadrightarrow (\aleph_{m+1}, \aleph_m)$

Theorem (E.-Hayut, 2018)

It is consistent relative to a huge cardinal that for all regular κ and all infinite $\mu < \kappa$, $(\kappa^+, \kappa) \twoheadrightarrow (\mu^+, \mu)$.

Theorem (Foreman, 1982)

For each n, it is consistent relative to a 2-huge cardinal that $(\aleph_{n+3}, \aleph_{n+2}, \aleph_{n+1}) \twoheadrightarrow (\aleph_{n+2}, \aleph_{n+1}, \aleph_n)$

Question (Foreman)

Is it consistent that for all $n > m$, $(\aleph_{n+2}, \aleph_{n+1}, \aleph_n) \twoheadrightarrow (\aleph_{m+2}, \aleph_{m+1}, \aleph_m)$?

Getting $(\kappa^{++},\kappa^+) \twoheadrightarrow (\kappa^+,\kappa)$

Theorem (Kunen, 1978)

κ is huge with target λ and $\mu < \kappa$ is regular, then there is a μ-closed forcing extension in which $\kappa = \mu^+$, $\lambda = \kappa^+$, and the hugeness embedding can be generically lifted, implying $(\mu^{++},\mu^+) \twoheadrightarrow (\mu^+,\mu)$ holds.

Kunen constructs a κ-c.c. forcing $\mathbb{P} \subseteq V_\kappa$ such that for many $\alpha < \kappa$, $\mathbb{P} \cap V_\alpha \trianglelefteq \mathbb{P}$ and $\mathbb{P} \cap V_\alpha * \mathbb{S}(\alpha,\kappa) \trianglelefteq \mathbb{P}$.

$\mathbb{S}(\alpha,\beta)$ is the Silver collapse, the collection of partial functions $p : \beta \times \alpha \to \beta$ such that $\mathrm{dom}(p) \subseteq X \times \xi$ for some $X \in [\beta]^{\leq\alpha}$ and $\xi < \alpha$, and for each $(\gamma,\delta) \in \mathrm{dom}(p)$, $p(\gamma,\delta) < \gamma$.

We will have $\mathbb{P} * \dot{\mathbb{S}}(\kappa,\lambda) \trianglelefteq j(\mathbb{P})$. If $G * H$ is generic, then we first lift to $j : V[G] \to M[G']$, with $H \in M[G']$.

For every $q \in H$, $\mathrm{dom}\, j(q) \in X \times \kappa$, for $X \in [j(\lambda)]^{\leq\lambda}$. Since $|H| = \lambda$, $\bigcup j``H \in \mathbb{S}(\lambda, j(\lambda))^{M[G']}$. Force below this to lift further to $j : V[G][H] \to M[G'][H']$.

Iteration and simplification

Foreman constructed a similar $\mathbb{P}$ that can be iterated, where $\mathbb{P}(\mu,\kappa) * \dot{\mathbb{P}}(\kappa,\lambda) \trianglelefteq \mathbb{P}(\mu,\lambda)$.

If we have $\kappa_1 < \kappa_2 < \kappa_3 < \dots$ huge cardinals that map to each other, then we can iterate $\mathbb{P}(\omega,\kappa_1) * \dot{\mathbb{P}}(\kappa_1,\kappa_2) * \dot{\mathbb{P}}(\kappa_2,\kappa_3) * \dots$ to get $(\omega_{n+1},\omega_n) \twoheadrightarrow (\omega_{m+1},\omega_m)$ for all $m < n < \omega$.

Kunen's and Foreman's constructions are somewhat complicated.

Definition (Shioya)

The Easton collapse $\mathbb{E}(\kappa,\lambda)$ is the Easton-support product of $\mathrm{Col}(\kappa,\alpha)$ over $\kappa \leq \alpha < \lambda$.

Theorem (E.)

$\mathbb{E}(\omega,\kappa_1) * \dot{\mathbb{E}}(\kappa_1,\kappa_2) * \dot{\mathbb{E}}(\kappa_2,\kappa_3) * \dots$ *forces that* $(\omega_{n+1},\omega_n) \twoheadrightarrow (\omega_{m+1},\omega_m)$ *holds for all* $m < n < \omega$.

Easton collapse

Lemma (McAloon?)

If $\mathbb{P}$ is κ-closed and collapses $|\mathbb{P}|$ to κ, then there is a dense embedding from $\mathsf{Col}(\kappa, |\mathbb{P}|)$ *to* $\mathbb{P}$.

Corollary

Assuming enough GCH, for regular $\mu < \kappa < \lambda$,
$\mathbb{E}(\mu, \lambda) \cong \mathbb{E}(\mu, \lambda) \times \mathbb{E}(\kappa, \lambda)$.

Proof:

$$\mathbb{E}(\mu, \lambda) \cong \mathbb{E}(\mu, \kappa) \times \prod_{\kappa \leq \alpha < \lambda}^{E} \mathsf{Col}(\mu, \alpha) \cong \mathbb{E}(\mu, \kappa) \times \prod_{\kappa \leq \alpha < \lambda}^{E} \mathsf{Col}(\mu, \alpha) \times \mathsf{Col}(\kappa, \alpha)$$

$$\cong \mathbb{E}(\mu, \kappa) \times \prod_{\kappa \leq \alpha < \lambda}^{E} \mathsf{Col}(\mu, \alpha) \times \mathbb{E}(\kappa, \lambda) \cong \mathbb{E}(\mu, \lambda) \times \mathbb{E}(\kappa, \lambda)$$

Easton collapse

If $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a forcing, the *termspace forcing* $T(\mathbb{P}, \dot{\mathbb{Q}})$ is the set of names for elements of $\mathbb{Q}$, ordered by $\dot{q}_1 \leq \dot{q}_0$ when $1 \Vdash \dot{q}_1 \leq \dot{q}_0$.

Lemma (Laver)

For all posets $\mathbb{P}$ and $\mathbb{P}$-names for posets $\dot{\mathbb{Q}}$, the identity map is a projection from $\mathbb{P} \times T(\mathbb{P}, \dot{\mathbb{Q}})$ to $\mathbb{P} * \dot{\mathbb{Q}}$.

Lemma

If $\mathbb{P}$ is κ-c.c. and of size κ, then for any $\delta \geq \kappa$ with $\delta^{<\kappa} = \delta$, there is a dense embedding from $\mathrm{Col}(\kappa, \delta)$ into $T(\mathbb{P}, \dot{\mathrm{Col}}(\kappa, \delta))$.

Corollary (Shioya)

If $\kappa < \lambda$ are Mahlo and $\mu < \kappa$ is regular, then there is a projection from $\mathbb{E}(\mu, \lambda)$ to $\mathbb{E}(\mu, \kappa) * \dot{\mathbb{E}}(\kappa, \lambda)$.

CC from Easton

Definition

We will say $\mathbb{P}$ is κ-*flat* when $\mathbb{P}$ can be written as an increasing union of regular suborders (filtration), $\mathbb{P} = \bigcup_{\alpha<\lambda} \mathbb{P}_\alpha$, where $\mathrm{cf}(\lambda) > \kappa$, each $\mathbb{P}_\alpha$ is κ-directed-closed with infima and of size $< \lambda$, there is a commuting system of continuous projections $\restriction \alpha : \mathbb{P} \to \mathbb{P}_\alpha$, and with the following property: Whenever $\langle p_\alpha : \alpha < \kappa \rangle \subseteq \mathbb{P}$ and $\langle \xi_\alpha : \alpha < \kappa \rangle \subseteq \lambda$ is increasing, and $p_\beta \restriction \xi_\alpha = p_\alpha$ for $\alpha < \beta < \kappa$, then $\langle p_\alpha : \alpha < \kappa \rangle$ has a lower bound in $\mathbb{P}$.

Note: $\mathbb{E}(\kappa, \lambda)$ is κ-flat.

Lemma

Suppose $j : M \to N$ is an elementary embedding between models of set theory and $\mathbb{P} \in M \cap N$ is κ-flat as witnessed by a filtration of length λ, $j(\kappa) = \lambda$, $j``\lambda \in N$, and there is $G \in N$ that is a $\mathbb{P}$-generic filter over $M$. Then $j``G$ has a lower bound in $j(\mathbb{P})$.

CC from Easton

Suppose κ is huge with target λ and $\mu < \kappa$ is regular. Let $G * H \subseteq \mathbb{E}(\mu, \kappa) * \dot{\mathbb{E}}(\kappa, \lambda)$ be generic. Let $G' \subseteq \mathbb{E}(\mu, \lambda)$ be such that $G * H$ is the projection of G'. First lift to $j : V[G] \to M[G']$.

In $V[G]$, $\mathbb{E}(\kappa, \lambda)$ is κ-flat with a filtration of length λ. Since $H \in M[G']$, the above lemma implies that $j``H$ has a lower bound in $E(\lambda, j(\lambda))^{M[G']}$.

Force below this to obtain H' and a lifting $j : V[G * H] \to M[G' * H']$. This shows $(\lambda, \kappa) \twoheadrightarrow (\kappa, \mu)$ holds in $V[G * H]$.

Triples strategy

Suppose μ is a regular cardinal, $\kappa > \mu$ is 2-huge, and $j : V \to M$ is a witnessing embedding with $j(\kappa) = \lambda$, $j(\lambda) = \theta$, and $M^\theta \subseteq M$. We want a μ-distributive forcing that makes $\kappa = \mu^+$, $\lambda = \kappa^+$, $\theta = \lambda^+$, and allows the embedding to be generically lifted.

The forcing will be of the form $P * Q * R$, where $P \subseteq V_\kappa$, $P * Q \subseteq V_\lambda$, and $P * Q * R \subseteq V_\theta$. We will need a projection $\pi_0 : j(P) \to P * Q$ with the following properties:

- The identity map is a complete embedding from P to $j(P)$, and $\pi_0 \restriction P = \text{id}$.
- Whenever $G' \subseteq j(P)$ is generic and $G * H = \pi_0[G']$, then $j[H]$ has a lower bound in $j(Q)^{M[G']}$.

Triples strategy

This will allow a lifting $j : V[G * H] \to M[G' * H']$ by forcing below such a lower bound. Further, we will need a $j(P)$-name for a condition $q^* \in j(Q)$ and projection $\pi_1 : j(P * Q) \restriction (1, q^*) \to P * Q * R$ such that:

- $\pi_1 \restriction j(P) = \pi_0$.
- If $G' \subseteq j(P)$ is generic and $G * H = \pi_0[G']$, q^* is forced to be a lower bound to $j[H]$.
- If $G' * H' \subseteq j(P * Q) \restriction (1, q^*)$ is generic and $G * H * K = \pi_1[G' * H']$, then $j[K]$ has a lower bound in $j(R)^{M[G' * H']}$.

Then we can force below a lower bound of $j[K]$ to lift the embedding through $G * H * K$.

Triples strategy

How can we achieve this? Since $j(P) \subseteq V_\lambda$, the work of absorbing a generic for R must be mostly the responsibility of $j(Q)$. R will be a relatively simple λ-closed θ-c.c. poset in V^{P*Q}, but it will no longer be λ-closed in $V^{j(P)}$. Over $V^{j(P)}$, it should be forced that $j(Q)$ absorbs such a poset R as constructed in the inner model V^{P*Q}.

Since $\kappa \notin \text{ran}(j)$, $j(Q)$ should include "versions" of R from a collection of inner models large enough to include M^{P*Q}, which will not be definable from paramters in the range of j. Therefore, $j(Q)$ should project to various such R as defined in different "cuts" of $j(P)$ into candidate factors $P_\alpha * Q_\alpha$, where $\mu < \alpha < \lambda$.

By elementarity, this requires that Q projects to many baby versions R_α of R contained in V_λ, as defined in various cuts of P into factors $P_\alpha * Q_\alpha$, where $\mu < \alpha < \kappa$. **Thus Q will not be κ-closed in V^P.**

Triples strategy

We should have a sequence of projections $\sigma_\alpha : P \to P_\alpha * Q_\alpha$ for appropriate cut points $\mu < \alpha < \kappa$, with $\pi_0 = j(\vec{\sigma})(\kappa)$. In order to find the appropriate master conditions, we will want to amalgamate local master conditions for the posets R_α as defined in $V^{P_\alpha * Q_\alpha}$, wherein R_α will be κ-directed-closed.

In order to amalgamate these local master conditions, we will want Q to project to versions of R in a well-organized way. For appropriate cut points α, $\mu < \alpha < \kappa$, we want a generic $G * H \subseteq P * Q$ to absorb a generic K_α for the version R_α as defined in $V^{P_\alpha * Q_\alpha}$ with the following property:

Suppose $G' \subseteq j(P)$ is generic. Let $G * H = \pi_0[G']$ and $G_\alpha * H_\alpha = \sigma_\alpha[G]$. Let $G_\alpha * H'_\alpha = j(\vec{\sigma})(\alpha)[G']$. Let $K_\alpha \subseteq R_\alpha$ be the generic absorbed in $V[G * H]$. **Then we want to also arrange that $K_\alpha \in V[G_\alpha * H'_\alpha]$.**

In this case, we can lift the embedding j to $j : V[G_\alpha * H_\alpha] \to V[G_\alpha * H'_\alpha]$, and $j[K_\alpha]$ will have a lower bound $r^*_\alpha \in j(R_\alpha)^{M[G_\alpha * H'_\alpha]}$.

Triples strategy

Then we will amalgamate the r^*_α into a sequence $r^* = \langle r^*_\alpha : \mu < \alpha < \kappa \rangle$. This will serve as a master condition for the part of Q that absorbs the versions R_α, which will essentially be a $<\kappa$-support product of these versions. If this suborder of Q is Q_0, and $H_0 = H \cap Q_0$, then we will be able to lift the embedding j to $j : V[G * H_0] \to M[G' * H'_0]$ by taking H'_0 generic with $r^* \in H'_0$. The quotient Q/Q_0 will be nice enough that lifting through the rest of H will be no trouble. This will require an extension of r^* to some q^*, below which we force to obtain $H' \subseteq j(Q)$.

So $(1, q^*)$ will serve as the desired master condition in $j(P * Q)$. By the way we will have set things up, $P * Q$ will be an appropriate cut of $j(P)$, and $M[G' * H']$ will possess a generic $K \subseteq R$, a poset which is λ-directed-closed in $V[G * H]$. A lower bound to $j[K]$ will exist, enabling a further lifting to $j[G * H * K] \to M[G' * H' * K']$.

Main forcing

For regular $\mu < \gamma$, we define a poset $P(\mu, \gamma)$ inductively. Let $P_0(\mu, \gamma) = \mathbb{E}(\mu, \gamma)$. For $n < \omega$ and a regular γ assume that we have defined $P_i(\alpha, \gamma)$ for $i \leq n$ and regular $\alpha < \gamma$. Define:

$$P_{n+1}(\mu, \gamma) = \prod_{\alpha \in (\mu,\gamma) \cap \mathsf{M}}^{\mathrm{E}} P_n(\alpha, \gamma)^{<\alpha}$$

Finally, let

$$P(\mu, \gamma) = \prod_{n \in \omega} P_n(\mu, \gamma)$$

M is the class of Mahlo cardinals. $P(\mu, \kappa)$ will be μ-directed-closed with infima and κ-c.c. for Mahlo κ.

Main forcing

Using the telescoping nature of $P(\mu,\kappa)$, for each $\alpha \in (\mu,\kappa) \cap \mathrm{M}$, there will be a projection

$$\chi_\alpha : P(\mu,\kappa) \to P(\alpha,\kappa)^{<\alpha}$$

For $\alpha \in (\mu,\kappa) \cap \mathrm{M}$, we send $p \mapsto \langle p(n)(\alpha) \rangle_{n>0}$, giving a map:

$$P(\mu,\kappa) \to \prod_{n>0} P_{n-1}(\alpha,\kappa)^{<\alpha} \cong P(\alpha,\gamma)^{<\alpha}$$

Next, for $\alpha \in (\mu,\kappa) \cap \mathrm{M}$, define

$$\bar{Q}(\mu,\alpha,\kappa) = P(\alpha,\kappa)^{<\alpha} \times \prod_{\beta \in (\alpha,\gamma) \cap \mathrm{M}}^{\mathrm{E}} P(\beta,\gamma)$$

along with a projection $\psi_\alpha : P(\mu,\kappa) \to P(\mu,\alpha) \times \bar{Q}(\mu,\alpha,\kappa)$, which is defined by:

$$\psi_\alpha(p) = (p \restriction \alpha, \chi_\alpha(p)^\frown \langle \chi_\beta(p)(\alpha) \rangle_{\beta \in (\alpha,\gamma) \cap \mathrm{M}})$$

Main forcing

For Mahlo $\alpha < \gamma \leq \kappa$, we inductively define

$$R(\mu, \alpha, \gamma) = P(\mu, \alpha) \star \left(\prod_{\zeta \in (\mu, \alpha) \cap M}^{\alpha} \dot{P}(\alpha, \gamma)^{R(\mu, \zeta, \alpha)} \times \prod_{\xi \in (\alpha, \gamma) \cap M}^{\mathrm{E}} \dot{P}(\xi, \gamma) \right)$$

along with projections $\varphi_{\alpha\gamma}$ from the posets $P(\mu, \alpha) \times \bar{Q}(\mu, \alpha, \gamma)$. This uses nested termspace projections.

Lifting argument

Suppose κ is huge with target λ and $\mu < \kappa$. If $G * H \subseteq R(\mu, \kappa, \lambda)$ is generic, then a further forcing gets a generic $G' \subseteq P(\mu, \lambda)$ that projects to $G * H$. We can lift to $j : V[G] \to M[G']$.

For each Mahlo $\alpha < \kappa$, there is a projection from G to a generic $G_\alpha * H_\alpha$ for $R(\mu, \alpha, \kappa)$. Also there is a projection from G' to $G_\alpha * H'_\alpha$, with $G_\alpha * H_\alpha$ as an initial segment. We can lift to $j : V[G_\alpha * H_\alpha] \to M[G_\alpha * H'_\alpha]$.

If $H(\alpha)$ is the α^{th} component of H, it is $P(\kappa, \lambda)$-generic over $V[G_\alpha * H_\alpha]$. If $H'_\alpha(\kappa)$ is the κ^{th} component of H'_α, it is $P(\kappa, \lambda)$-generic over $V[G_\alpha]$.

The termspace projection will yield a filter H'' from $H'_\alpha(\kappa)$, computable in $M[G_\alpha * H_\alpha]$ that is $P(\kappa, \lambda)$-generic over $V[G_\alpha * H_\alpha]$. It turns out by the way the maps are defined that $H'' = H(\alpha)$. Since $P(\kappa, \lambda)$ is κ-flat, there is a condition $r_\alpha \in M[G_\alpha * H'_\alpha]$ that is below $j``H(\alpha)$.

Lifting argument

Let $r = \langle r_\alpha : \alpha < \kappa \rangle$. Using κ-flatness again, there is s that is below $H \restriction (\kappa, \lambda)$. Forcing below $r^\frown s$ allows a lifting to $j : V[G \star H] \to M[G' \star H']$.

Now suppose the embedding was moreover 2-huge, with $j(\lambda) = \theta$. The κ^{th} component of H' is a filter K that is $P(\lambda, \theta)$-generic over $V[G \star H]$. Adjoining this and invoking λ-flatness allows a further generic lift to

$$j : V[G \star H][K] \to M[G' \star H'][K']$$

We conclude that $(\theta, \lambda, \kappa) \twoheadrightarrow (\lambda, \kappa, \mu)$ holds in $V[G \star H][K]$.

Iterating

The forcing used was $(P(\mu, \kappa) \star Q(\kappa, \lambda)) * \dot{P}(\lambda, \theta)$. If λ is itself 2-huge, we can continue with the next Q. If we have an ω-chain of 2-huge cardinals $\kappa_1 < \kappa_2 < \kappa_3 < \dots$, then a general preservation argument shows that

$$(P(\omega, \kappa_1) \star Q(\kappa_1, \kappa_2)) * (\dot{P}(\kappa_2, \kappa_3) \star \dot{Q}(\kappa_3, \kappa_4)) * (\dot{P}(\kappa_4, \kappa_5) \star \dots$$

forces $(\omega_{n+3}, \omega_{n+2}, \omega_{n+1}) \twoheadrightarrow (\omega_{n+2}, \omega_{n+1}, \omega_n)$ to hold for all even $n < \omega$.

Even more triples

Modifying P slightly and the construction of R more seriously, we can get another version of $P(\mu, \kappa)$ where for all selections of an odd number of Mahlo cardinals $\mu < \alpha_1 < \cdots < \alpha_n < \kappa$, we get a projection of $P(\mu, \kappa)$ to a poset of the form

$$A_{\langle \mu, \alpha_1, \ldots, \alpha_n, \kappa \rangle} = (P(\mu, \alpha_1) \star Q(\alpha_1, \alpha_2)) * \cdots * (\dot{P}(\alpha_{n-1}, \alpha_n) \star \dot{Q}(\alpha_n, \kappa))$$

And for any Mahlo $\gamma < \kappa$, there is a projection from $P(\mu, \kappa)$ to a poset of the form $P(\mu, \gamma) \star Q(\gamma, \kappa)$, where $Q(\gamma, \kappa)$ absorbs the posets $P(\gamma, \kappa)$ as defined in all the subextensions $A_{\langle \mu, \alpha_1, \ldots, \alpha_n, \gamma \rangle}$ of $P(\mu, \gamma)$.

Another lifting

Lemma

If θ is 3-huge, then there is a sequence $\kappa_1 < \kappa_2 < \kappa_3 < \dots$ below θ such that for every $m < n$, there is an embedding $j : V \to M$ such that $j(\kappa_{m+i}) = \kappa_{n+i}$ for $i < 2$, and M is closed under κ_{n+2}-sequences.

So now let us consider forcing with $(P(\omega, \kappa_1) \star Q(\kappa_1, \kappa_2)) * (\dot{P}(\kappa_2, \kappa_3) \star \dot{Q}(\kappa_3, \kappa_4))$. We want to show that $(\omega_4, \omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1, \omega)$ holds. Take an embedding as above that sends κ_n to κ_{n+2} for $n = 1, 2$.

$j(P(\omega, \kappa_1)) = P(\omega, \kappa_3)$, and it absorbs $(P(\omega, \kappa_1) \star Q(\kappa_1, \kappa_2)) * \dot{P}(\kappa_2, \kappa_3)$. We can easily lift through the first stage, say $j : V[G_1] \to M[G_1']$, and we will get projected $G_2, G_3 \in M[G_1']$ for $Q(\kappa_1, \kappa_2)$ and $P(\kappa_2, \kappa_3)$ respectively.

With similar arguments as before, we can extend this to lift further to $j : V[G_1, G_2, G_3] \to M[G_1', G_2', G_3']$.

Another lifting

After forcing up to $P(\kappa_2, \kappa_3)$, $Q(\kappa_3, \kappa_4)$ absorbs the versions of $P(\kappa_3, \kappa_4)$ as defined in all the intermediate extensions by $(P(\omega, \kappa_1) \star Q(\kappa_1, \kappa_2)) * A_{\langle \kappa_2, \vec{\alpha}, \kappa_3 \rangle}$.

In $M[G']$, $j(Q(\kappa_1, \kappa_2))$, which is the $Q(\kappa_3, \kappa_4)$ after forcing with $P(\omega, \kappa_3)$, absorbs the versions of $P(\kappa_3, \kappa_4)$ as defined in all the intermediate extensions by $A_{\langle \omega, \vec{\alpha}, \kappa_3 \rangle}$. But this is a superset of the ones absorbed by the $Q(\kappa_3, \kappa_4)$ of $(P(\omega, \kappa_1) \star Q(\kappa_1, \kappa_2)) * \dot{P}(\kappa_2, \kappa_3)$.

Thus G_2' also absorbs a generic G_4 for the smaller $Q(\kappa_3, \kappa_4)$.

Using a flatness argument, there will be a condition below $j``G_4$, so we can lift further to include $V[G_1, G_2, G_3, G_4]$, obtaining $(4, 3, 2) \twoheadrightarrow (2, 1, 0)$.

Iterating

Some preservation arguments will show that after doing this ω-times will full support, for all even n, the model will satisfy

$$(n+3, n+2, n+1) \twoheadrightarrow (n+2, n+1, n)$$

and

$$(n+4, n+3, n+2) \twoheadrightarrow (n+2, n+1, n).$$

Using the transitivity of this principles, we get

$$(m+2, m+1, m) \twoheadrightarrow (n+2, n+1, n)$$

for all even n and all $m > n$.