Ramsey Theory on Infinite Structures, Part III

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Young Set Theory, Münster, 2023

Grateful for research support from NSF grant 1901753

Day 3: Infinite-dimensional Ramsey theory

I. Infinite-dimensional Ramsey Theory on ω.
   (a) Proofs using combinatorial forcing.

II. Topological Ramsey Spaces.
   (a) Definitions.
   (b) The Four Axioms and Abstract Ellentuck Theorem.
   (c) Examples.

III. Infinite-dimensional Structural Ramsey Theory.
   (a) Extending big Ramsey degree results.
   (b) Using forcing to prove Pigeonholes (Axiom A.4).

IV. More Directions and Open Problems.

V. References.

I. Infinite-dimensional Ramsey Theory on ω.

Ramsey subsets of the Baire space

A subset $\mathcal{X}$ of $[\omega]^\omega$ is **Ramsey** if each for $M \in [\omega]^\omega$, there is an $N \in [M]^\omega$ such that $[N]^\omega \subseteq \mathcal{X}$ or $[N]^\omega \cap \mathcal{X} = \emptyset$.

Ramsey's Theorem (topological form). For any m and r, if $\mathcal{X} \subseteq [\omega]^\omega$ is a union of basic clopen sets of the form $[s, \omega]$ where $s \in [\omega]^m$, then $\mathcal{X}$ is Ramsey.

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AC $\Rightarrow \exists \mathcal{X} \subseteq [\omega]^\omega$ which is not Ramsey.

Solution: restrict to 'definable' sets.

Infinite-dimensional Ramsey Theory

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Nash-Williams Thm. Clopen sets are Ramsey.

Galvin–Prikry Thm. Borel sets are Ramsey.

Silver Thm. Analytic sets are Ramsey.

Ellentuck Thm. A set is completely Ramsey iff it has the property of Baire in the Ellentuck topology.

Ellentuck Theorem

Ellentuck topology: refines the metric topology with basic open sets

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Theorem (Ellentuck)

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$$(*) \qquad \forall [s, A] \;\; \exists B \in [s, A] \text{ such that } [s, B] \subseteq \mathcal{X} \text{ or } [s, B] \cap \mathcal{X} = \emptyset$$

iff $\mathcal{X}$ has the property of Baire with respect to the Ellentuck topology.

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The Ellentuck space is the prototype for **topological Ramsey spaces**:

Ellentuck Theorem

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$(*)$ is called completely Ramsey by Galvin–Prikry and Ramsey by Todorcevic.

The Ellentuck space is the prototype for **topological Ramsey spaces**: Points are infinite sequences, topology is induced by finite heads and infinite tails, and every subset with the property of Baire satisfies $(*)$.

Nash-Williams Theorem

Definition

A family $\mathcal{F} \subseteq [\omega]^{<\omega}$ is **Nash-Williams** iff $s \neq t$ in $\mathcal{F}$ implies $s \not\sqsubseteq t$.

Definition

$\mathcal{F} \subseteq [\omega]^{<\omega}$ is **Ramsey** iff for each partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$, there is an $M \in [\omega]^{\omega}$ such that $\mathcal{F}_i | M = \emptyset$.

Theorem (Nash-Williams)

Every Nash-Williams family is Ramsey.

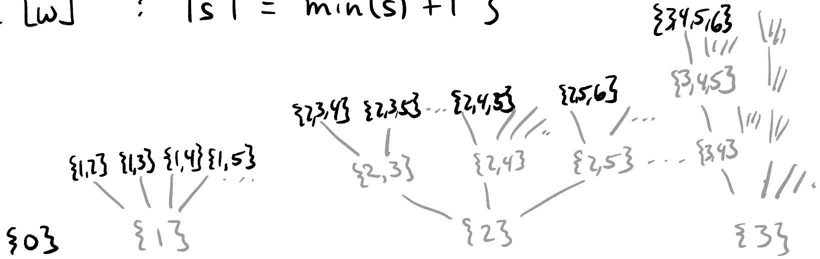
Proof developed "combinatorial forcing".

Theorem (Nash-Williams)

Every Nash-Williams family is Ramsey.

Example: Schreier Barrier

$$\{ s \in [\omega]^{<\omega} : |s| = \min(s) + 1 \}$$



Galvin-Prikry Theorem

Theorem (Galvin-Prikry)

Every Borel set $\mathcal{X} \subseteq [\omega]^\omega$ satisfies

$$\forall [s, A] \;\; \exists B \in [s, A] \text{ such that } [s, B] \subseteq \mathcal{X} \text{ or } [s, B] \cap \mathcal{X} = \emptyset.$$

Proof uses combinatorial forcing to show that "Every open set is Ramsey."

Def: $\mathcal{X} \subseteq [\omega]^\omega$ is Completely Ramsey (CR) if this line holds.

The rest of the proof has the following outline:

Galvin-Prikry Theorem

I. Every open set is CR.

II. Complements of CR sets are CR.

III. If $\mathcal{X}$ is CR, $A \in [\omega]^\omega$, and $s \sqsubset A$, then $\exists B \in [s, A]$ s.t. $\mathcal{X} \cap [A]^\omega$ is open in the subspace topology. (Ellentuck took this one step further and used $[s, A]$ as a top.)

IV. The countable union of CR sets is CR.

<u>Conclude</u>: Borel sets are CR!

Ellentuck Theorem

Theorem (Ellentuck)

A set $\mathcal{X} \subseteq [\omega]^\omega$ satisfies

$$\forall [s, A] \ \ \exists B \in [s, A] \text{ such that } [s, B] \subseteq \mathcal{X} \text{ or } [s, B] \cap \mathcal{X} = \emptyset$$

iff $\mathcal{X}$ has the property of Baire with respect to the Ellentuck topology.

A set $\mathcal{X}$ has the property of Baire

$\iff \mathcal{X} = \mathcal{O} \triangle \mathcal{M}$

for some open set $\mathcal{O}$ and some meager set $\mathcal{M}$.

Note: $s = \emptyset$ gives $\omega \longrightarrow (\omega)^\omega$. Holds in $L(\mathbb{R})$, and under $AD_{\mathbb{R}}$, $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$.

Ellentuck Theorem

Ellentuck's proof closely follows Galvin-Prikry, with an important tweak.

Here, we follow the proof of Thm 1.54 in Todorcevic's book.

Fix $\mathcal{X} \subseteq [\omega]^\omega$. $s, t, u, \ldots \in [\omega]^{<\omega}$, $A, B, C, \ldots \in [\omega]^\omega$.

Def: A accepts s if $[s, A] \subseteq \mathcal{X}$.

A rejects s if $\forall B \subseteq A$, B does not accept s.

A decides s if either A accepts s or A rejects s.

(copy on board)

Lemma 1: (a) Accepting and rejecting are preserved under $\supseteq$.

(b) $\forall s\ \forall A$, $\exists B \subseteq A$ which decides s.

Lemma 2: $\forall A\ \exists B \subseteq A$ s.t. B decides all $s \in [B]^{<\omega}$.

Pf: Take $A_0 \subseteq A$ deciding $\emptyset$. Let $b_0 = \min(A_0)$.
Take $A_1 \subseteq A_0 \setminus \{b_0\}$ deciding $\{b_0\}$. Let $b_1 = \min(A_1)$.
In 2 steps, take $A_2 \subseteq A_1 \setminus \{b_1\}$ deciding both $\{b_1\}$ and $\{b_0, b_1\}$. (recall pf of RT on Day 1)
Let $b_2 = \min(A_2)$.

Ellentuck Theorem

For the inductive step, given A_n and $b_n = \min(A_n)$, enumerate all subsets of $\{b_0, b_1, \dots, b_n\}$ containing b_n. Find $A_{n+1} \subseteq A_n \setminus \{b_n\}$ deciding all of them.

⋮

Let $B = \{b_i : i < \omega\}$.

Claim: $\forall s \in [B]^{<\omega}$, B decides s.

☒

(This is a very common type of argument in tRs's)

Ellentuck Theorem

Lemma 3: Suppose A decides all of its finite sets. If A rejects s, then A rejects $s \cup \{n\}$ $\forall^\infty n \in A$.

Lemma 4: Suppose A decides all of its finite sets. If A rejects ϕ, then $\exists B \subseteq A$ s.t. B rejects each $s \in [B]^{<\omega}$.

Pf Idea: Repeated application of Lemma 3 on finite sets with fixed max.

Ellentuck Theorem

<u>Lemma 5</u>: Let $\mathcal{O}$ be Ellentuck open subset of $[\omega]^\omega$.
Then $\forall$ basic open $[s, A]$, $\exists B \in [s, A]$ s.t.
either $[s, B] \subseteq \mathcal{O}$ or $[s, B] \cap \mathcal{O} = \phi$.

Pf Idea: Apply Lemmas 1-4 relativized to $[s, A]$.

(Replace ϕ by s.)

If A $\mathcal{O}$-accepts s, done.
Otherwise, Lemma 4 $\Longrightarrow$ $\exists B \in [s, A]$ that $\mathcal{O}$-rejects all $t \sqsupseteq s$ with $t \subseteq B$. Then $[s, B] \cap \mathcal{O} = \phi$.

$\boxtimes$

Ellentuck Theorem

Lemma 6: Let $\mathcal{M}$ be an Ellentuck-meager set. Then $\forall [s,A]\ \exists B \in [s,A]$ s.t. $[s,B] \cap \mathcal{M} = \phi$.

Pf Idea: $\mathcal{M} = \bigcup_{n<\omega} \mathcal{N}_n$ for some nowhere dense sets $\mathcal{N}_n$.

Note: $\forall n$, Lem 5 $\Longrightarrow \forall [s,C]\ \exists D \in [s,C]$ s.t. $[s,D] \cap \mathcal{N}_n = \phi$, since $\overline{\mathcal{N}_n}$ is n.d. & has open complement.

Now do a diagonalization.

Ellentuck Theorem

To finish the proof of Ellentuck's Theorem,

Let $\mathcal{O}$ be open and $\mathcal{M}$ be meager s.t. $\mathcal{X} = \mathcal{O} \triangle \mathcal{M}$.

Then $\mathcal{X} \triangle \mathcal{O} = \mathcal{M}$.

Lem 6 $\Rightarrow \exists B \in [s, A]$ s.t.

$[s, B] \cap \mathcal{M} = \emptyset$.

Lem 5 $\Rightarrow \exists C \in [s, B]$ s.t.

$[s, C] \subseteq \mathcal{O}$ or $[s, C] \cap \mathcal{O} = \emptyset$. ☒

II. Topological Ramsey Spaces

II(a). Topological Ramsey Spaces

History:

Carlson and Carlson-Simpson 1980's and 1990's.

Todorcevic Book 2010.

$$(\mathcal{R}, \leq, r)$$

for some $n < \omega$

$$[a, B] = \{A \in \mathcal{R} : a = r_n(A) \wedge A \leq B\}$$

Definition

A triple $(\mathcal{R}, \leq, r)$ is a **topological Ramsey space** if every subset with the property of Baire is Ramsey and every meager subset is Ramsey null.

The following 4 Axioms guarantee that a space behaves like the Ellentuck space.

These guarantee infinite-dimensional Ramsey Theorems of the form $A \to (A)^A$ where $A \in \mathcal{R}$, an injective tRS $\mathcal{R}$, in models of ZF where all subsets of $\mathcal{R}$ are sufficiently definable.

Todorcevic's Axioms for Topological Ramsey Spaces

$(\mathcal{R}, \leq, r)$. $\mathcal{AR} = \{r_n(A) : A \in \mathcal{R} \wedge n < m\}$

A.1 (Sequencing)

(1) $r_0(A) = \emptyset$ for all $A \in \mathcal{R}$,

(2) $B \neq A$ implies that $r_n(A) \neq r_n(B)$ for some n,

(3) $r_m(A) = r_n(B)$ implies $m = n$ and $r_k(A) = r_k(B)$ for all $k \leq m$.

A.2 (Finitization) There is a transitive, reflexive relation $\leq_{\text{fin}}$ on $\mathcal{AR}$ such that

(1) $\{a \in \mathcal{AR} : a \leq_{\text{fin}} b\}$ is finite for all $b \in \mathcal{R}$,

(2) $A \leq B$ iff $\forall m\, \exists n$ such that $r_m(A) \leq_{\text{fin}} r_n(B)$,

(3) $\forall a, b \in \mathcal{AR}\, [a \sqsubseteq b$ and $b \leq_{\text{fin}} c \rightarrow \exists d \sqsubseteq c\ a \leq_{\text{fin}} d]$.

Example: Ellentuck space.

Todorcevic's Axioms for Topological Ramsey Spaces

A.3 (Amalgamation)

(1) $\forall a \in \mathcal{AR}\ \forall B \in \mathcal{R}$,

$$d = \mathrm{depth}_B(a) < \infty \ \rightarrow\ \forall A \in [d, B]\ ([a, A] \neq \emptyset),$$

(2) $\forall a \in \mathcal{AR}\ \forall A, B \in \mathcal{R}$, letting $d = \mathrm{depth}_B(a)$,

$$A \leq B \text{ and } [a, A] \neq \emptyset\ \rightarrow\ \exists C \in [d, B]\ ([a, C] \subseteq [a, A]).$$

A.4 (Pigeonhole) Suppose $a \in \mathcal{AR}_k$ and $\mathcal{O} \subseteq \mathcal{AR}_{k+1}$. Then for every $B \in \mathcal{R}$ such that $[a, B] \neq \emptyset$, there exists $A \in [r_K(B), B]$, where $d = \mathrm{depth}_B(a)$, such that the set $\{r_{k+1}(C) : C \in [a, A]\}$ is either contained in $\mathcal{O}$ or is disjoint from $\mathcal{O}$.

Example: Ellentuck space.

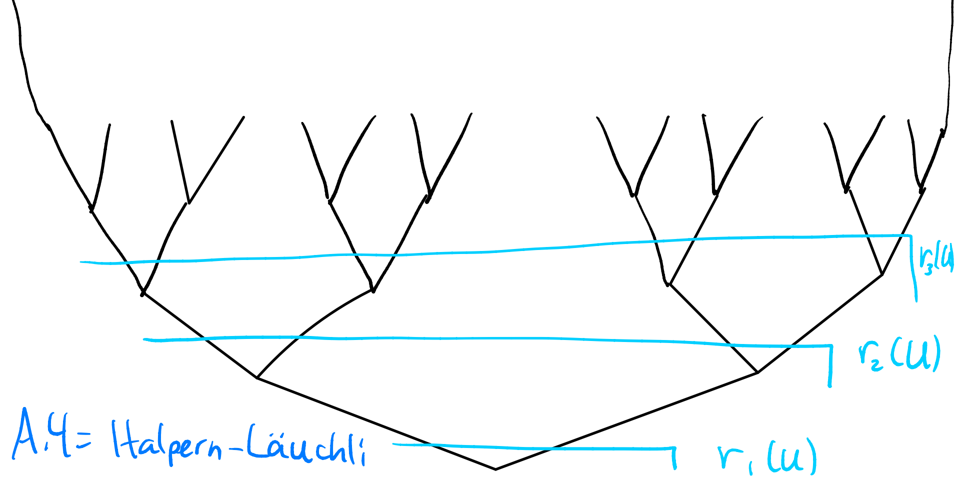
II(c). Examples of Topological Ramsey Spaces

- Ellentuck space
- Milliken strong trees
- $\mathrm{FIN}^{[\infty]}$
- Many more.

II(c). Milliken strong trees (1981)

$\mathcal{AR}_n = \mathcal{S}_n(\mathbb{U})$ = set of n-strong subtrees.

$[s, T]$ = {strong subtrees of T end-extending s}.

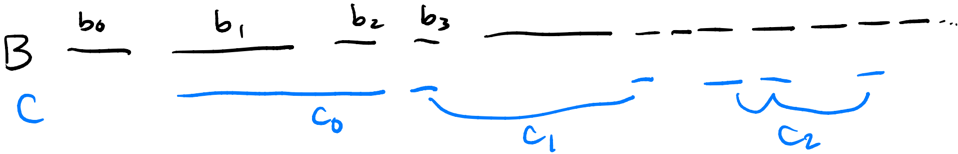


A.4 = Halpern-Läuchli

II(c). Milliken's block sequence space $\mathrm{FIN}^{[\infty]}$ (1975)

Maximal seq: $A = \{\{n\} : n \in \omega\}$

$B \in \mathrm{FIN}^{[\infty]}$ means $B = \langle b_n : n \in \omega \rangle$ each $b_n \in \mathrm{FIN}$ and $b_0 < b_1 < b_2 < \dots$



$C \leq B \Rightarrow$ the blocks in C are finite unions of blocks in B.

A.4 = Hindman's Theorem

For more on (topological) Ramsey spaces, see Todorcevic's 2010 book, *Introduction to Ramsey spaces*.

III. Infinite-dimensional Structural Ramsey Theory

KPT Question

Problem 11.2 in [KPT 2005]. Given a homogeneous structure $\mathbf{K}$, find the right notion of 'definable set' so that all definable subsets of $\binom{\mathbf{K}}{\mathbf{K}}$ are Ramsey.

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We assume the universe of $\mathbf{K}$ is ω so that $\binom{\mathbf{K}}{\mathbf{K}}$ is a subspace of $[\omega]^\omega$.

Constraint: Big Ramsey degrees.

Must fix a big Ramsey structure and work on subcopies (embeddings) of it.

The **right** theorem should directly recover exact big Ramsey degrees.

Infinite-Dimensional Ramsey Theory for the Rado graph

Theorem (D. 2019)

Fix an enumeration of the Rado graph and let U be its coding tree. Then the space of all subcopies of that coding tree has the property that all Borel sets are Ramsey.

Funnily, even though coding trees and forcing on them were developed to handle BRD of $\mathcal{H}_3$ - forbidden substructures - they turned out to be useful for developing ∞-dim'l structural R.T.

Infinite-Dimensional Ramsey Theory for the Rado graph

Galvin - Prikry analogue

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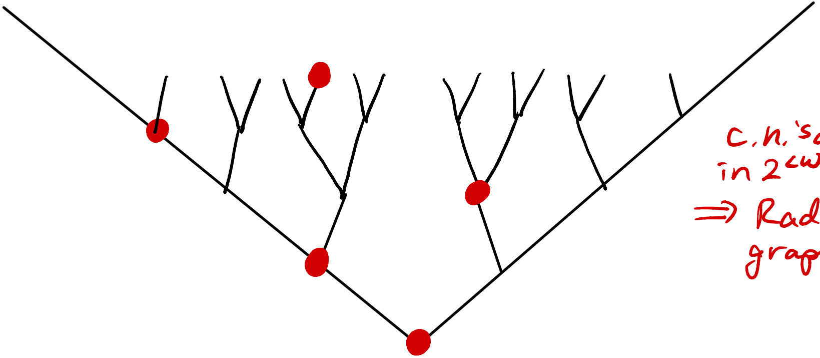


c.n.'s dense in $2^{<\omega}$
$\Rightarrow$ Rado graph

Fix an enumerated Rado graph $\mathbb{R}$. Let S be its coding tree.

Let $\mathcal{R}$ be the set of all subtrees of S which code $\mathbb{R}$ in the same way as S.

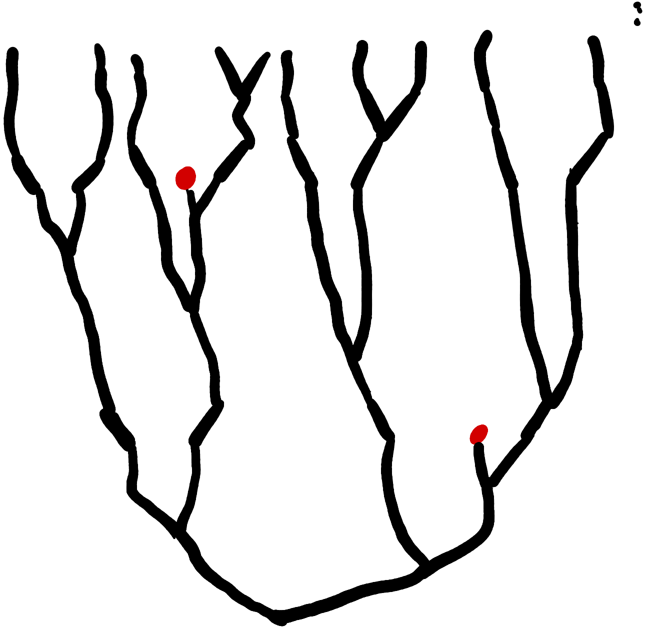
Finitization map: $r_n(T) = 1^{st}$ n levels of T.

$[a, T]$ = all $U \subseteq T$ in $\mathcal{R}$ end-extending a.

Thm implies $S \xrightarrow{*} (S)^S$ * for all Borel subsets.

This Theorem, however did not directly recover exact BRD.

Recall 'diaries' = diagonal antichain plus possibly more



Infinite-Dimensional Ramsey Theory for SDAP$^+$ structures

Theorem (D. 2022)

Let $\mathbf{K}$ *be a Fraïssé structure satisfying SDAP*$^+$ *with finitely many relations of arity at most two. Let* Δ *be a good diary representing* $\mathbf{K}$*. Then every Borel subset of* $\mathcal{R}(\Delta)$ *is completely Ramsey.*

Examples: Rado graph, k-partite graphs, ordered versions.

Proof follows Galvin-Prikry but uses forcing for a stronger Pigeonhole and a new style of combinatorial forcing.

Corollary

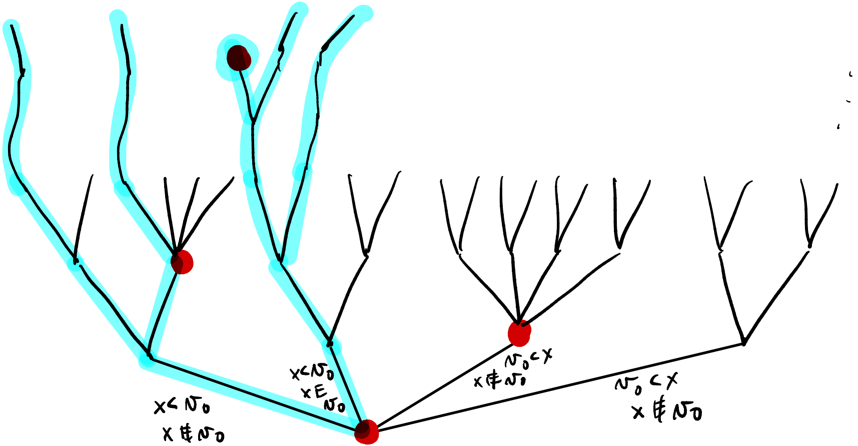
If $\mathbf{K}$ *has a certain amount of rigidity, Axiom A.3(2) of Todorcevic also holds, so we obtain analogues of Ellentuck's Theorem.*

Examples: The rationals, $\mathbb{Q}_n$, $\mathbb{Q}_{\mathbb{Q}}$.

Infinite-Dimensional Ramsey Theory for SDAP$^+$ structures

ordered Rado graph

Can pick out a diagonal antichain still representing the ordered Rado graph



We wanted to see if we could get a stronger ∞-dimensional theorem for the Rado graph, and also extend to k-clique-free graphs and FAP more generally.

Theorem (D.–Zucker)

Fix a finitely constrained binary free amalgamation class $\mathcal{K}$ *and let* $\mathbf{K} = Flim(\mathcal{K})$. *Then* $\mathbf{K}$ *has infinite-dimensional Ramsey theory which directly recovers exact big Ramsey degrees in (BCDHKVZ 2021).*

The strength of the theorem ranges from 'Souslin-measurable sets are Ramsey' (more than a Silver theorem analogue) to an analogue of the Ellentuck Theorem.

Abstract Ramsey Theorem

Theorem (Todorcevic)

Suppose that $(\mathcal{R}, \mathcal{S}, \leq, \leq_{\mathcal{R}})$ with finite restriction maps satisfying axioms **A.1–A.4**, *and that $\mathcal{S}$ is closed. Then the field of $\mathcal{S}$-Ramsey subsets of $\mathcal{R}$ is closed under the Souslin operation and it coincides with the field of $\mathcal{S}$-Baire subsets of $\mathcal{R}$.*

When $\mathcal{R} = \mathcal{S}$, this theorem implies the Abstract Ellentuck Theorem.

Theorem (D.–Zucker)

The conclusion of the above theorem still holds when axiom **A.3(2)** *is replaced by the weaker existence of an* **A.3(2)***-ideal.*

$\mathcal{S}$-Baire and $\mathcal{S}$-Ramsey sets

For $X \in \mathcal{S}$ and a a finite approximation to some member of $\mathcal{R}$,

$$[a, X] = \{A \in \mathcal{R} : A \leq_{\mathcal{R}} X \text{ and } a \sqsubset A\}$$

A set $\mathcal{X} \subseteq \mathcal{R}$ is **$\mathcal{S}$-Baire** if for every non-empty basic open set $[a, X]$ there is an $a \sqsubseteq b \in \mathcal{AR}$ and $Y \leq X$ in $\mathcal{S}$ such that $[b, Y] \neq \emptyset$ and $[b, Y] \subseteq \mathcal{X}$ or $[b, Y] \subseteq \mathcal{X}^c$.

$\mathcal{S}$-Ramsey requires $b = a$ and $Y \in [\text{depth}_X(a), X]$.

Axioms for Ramsey Spaces

$(\mathcal{R}, \mathcal{S}, \leq, \leq_{\mathcal{R}})$ and finite restrictions maps;
$\leq \subseteq \mathcal{S} \times \mathcal{S}$ and $\leq_{\mathcal{R}} \subseteq \mathcal{R} \times \mathcal{S}$.

A.1 (Sequencing) For any choice of $\mathcal{P} \in \{\mathcal{R}, \mathcal{S}\}$,

1. $M|_0 = N|_0$ for all $M, N \in \mathcal{P}$,
2. $M \neq N$ implies that $M|_n \neq N|_n$ for some n,
3. $M|_m = N|_n$ implies $m = n$ and $M|_k = N|_k$ for all $k \leq m$.

A.2 (Finitization) There is a transitive, reflexive relation $\leq_{\mathrm{fin}} \subseteq \mathcal{AS} \times \mathcal{AS}$ and a relation $\leq_{\mathrm{fin}}^{\mathcal{R}} \subseteq \mathcal{AR} \times \mathcal{AR}$ which are finitizations of the relations $\leq$ and $\leq_{\mathcal{R}}$, meaning that the following hold:

1. $\{a : a \leq_{\mathrm{fin}}^{\mathcal{R}} x\}$ and $\{y : y \leq_{\mathrm{fin}} x\}$ are finite for all $x \in \mathcal{S}$,
2. $X \leq Y$ iff $\forall m\, \exists n$ such that $X|_m \leq_{\mathrm{fin}} Y|_n$,
3. $A \leq_{\mathcal{R}} X$ iff $\forall m\, \exists n$ such that $A|_m \leq_{\mathrm{fin}}^{\mathcal{R}} X|_n$,
4. $\forall a \in \mathcal{AR}\ \forall x, y \in \mathcal{AS}\ [a \leq_{\mathrm{fin}}^{\mathcal{R}} x \leq_{\mathrm{fin}} y \rightarrow a \leq_{\mathrm{fin}}^{\mathcal{R}} y]$,
5. $\forall a, b \in \mathcal{AR}\ \forall x \in \mathcal{AS}\ [a \sqsubseteq b$ and $b \leq_{\mathrm{fin}}^{\mathcal{R}} x \rightarrow \exists y \sqsubseteq x\ a \leq_{\mathrm{fin}}^{\mathcal{R}} y]$.

Todorcevic's Axioms 3 and 4 for Ramsey Spaces

A.3 (Amalgamation)

(1) $\forall a \in \mathcal{AR}\ \forall Y \in \mathcal{S}$,

$$[d = \mathrm{depth}_Y(a) < \infty \ \rightarrow\ \forall X \in [d, Y]\ ([a, X] \neq \emptyset)],$$

(2) $\forall a \in \mathcal{AR}\ \forall X, Y \in \mathcal{S}$, letting $d = \mathrm{depth}_Y(a)$,

$$[X \leq Y \text{ and } [a, X] \neq \emptyset \ \rightarrow\ \exists Y' \in [d, Y]\ ([a, Y'] \subseteq [a, X])].$$

A.4 (Pigeonhole) Suppose $a \in \mathcal{AR}_k$ and $\mathcal{O} \subseteq \mathcal{AR}_{k+1}$. Then for every $Y \in \mathcal{S}$ such that $[a, Y] \neq \emptyset$, there exists $X \in [Y|_d, Y]$, where $d = \mathrm{depth}_Y(a)$, such that the set $\{A|_{k+1} : A \in [a, X]\}$ is either contained in $\mathcal{O}$ or is disjoint from $\mathcal{O}$.

A.3(2)-ideals

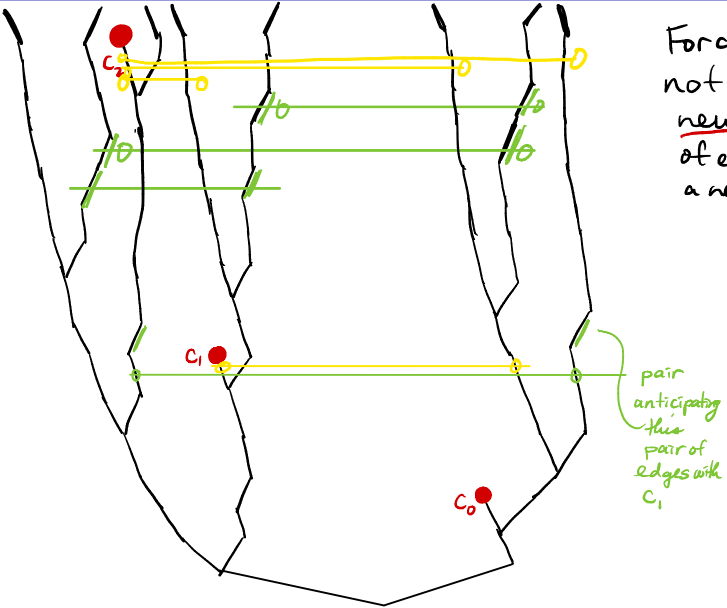
An ideal $\mathcal{I} \subseteq \mathcal{S} \times \mathcal{S}$ is a set satisfying

- $(X, Y) \in \mathcal{I} \Rightarrow X \leq Y$.
- $(X, Y) \in \mathcal{I}$ and $Z \leq X \Rightarrow (Z, Y) \in \mathcal{I}$.

$\mathcal{I}$ is an **A.3(2)-ideal** if additionally

- $\forall Y \in \mathcal{S}\ \forall n < \omega\ \exists Y' \in \mathcal{S}$ with $(Y', Y) \in \mathcal{I}$ and $Y'|_n = Y|_n$.
- If $(X, Y) \in \mathcal{I}$ and $a \in \mathcal{AR}^{\mathcal{X}}$, there is $Y' \in \mathcal{S}$ with $Y' \in [\mathrm{depth_Y(a), Y}]$, $(Y', Y) \in \mathcal{I}$, and $[a, Y'] \subseteq [a, X]$.

Question. Are A.3(2)-ideals necessary?



Forcing must not add new pairs of edges with a new vertex.

[redacted] Diaries and Forcing A.4

The forcing produces a Halpern-Läuchli style theorem, but keeping in mind the

a) coding nodes & by default yellow bits

b) splitting nodes

c) green lines

d) not adding new bits of forbidden substructures

IV. More Directions

- Non-forcing proofs.
- Higher arities.
- Infinite-dimensional structural Ramsey theory.
- Computability Theory and Reverse Mathematics.
- Topological dynamics correspondence.
- When exactly does $\mathcal{K}$ having small Ramsey degrees imply Flim($\mathcal{K}$) has finite big Ramsey degrees?
- What amalgamation or other properties of $\mathcal{K}$ correspond to the characterization of its big Ramsey degrees?

IV. Open Problems

1) Exact big Ramsey degrees for all Fraïssé classes which have small Ramsey degrees, and a language with finitely many relations of any given arity.
 — especially ternary relations and above

2) Does finite big Ramsey degrees always imply ∃ a big Ramsey structure? (Zucker)

3) Topological dynamics correspondence to BRD & ∞-diml structural RT?

IV. Open Problems

4) Ellentuck or other ∞-dim'l RT for poset w/l.o.,
k-regular hypergraphs,
all (ordered) FAP classes? (since they have RP)
Tournaments with certain forbidden tournaments (Sauer)

5) tRs's, ultrafilters, forcing connections.
See (2021) reference and Yuan Yuan Zheng's work, RK, Tukey, Sacks preserves certain uf's.

6) Ramsey spaces and $AD_{\mathbb{R}}$ or $L(\mathbb{R})$, etc.
See D-Hathaway (2021) extending Henle-Mathias-Woodin (1985) "Barren extensions"

b) Uncountable realm: Shelah 282: Con(HL(κ), κ mbl)

Džamonja-Larson-Mitchell: bRD of κ-rationals and κ-Rado graph at κ 2009 Israel JM
2009 AFML

Jing Zhang 2019, Tail-cone RT at κ mbl and analogue of Laver's Thm

D-Hathaway - lowering upper bd on consistency of HL(κ) at mbl (JSL 2017) and preservation via small forcings (JSL 2020)

D-Shelah 2022 arxiv.

Many open problems here on HL at large cards.

Expository References

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Although Harrington's proof is written better in JML2023.

Some of the many other References

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Todorcevic, *Introduction to Ramsey Spaces*, 2010.

Thank you very much!

Thank you very much!

Go prove some cool theorems!