Ramsey Theory on Infinite Structures

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Day 1: Classic Methods for $\mathbb{Q}$ and Forcing Halpern-Läuchli

I. Ramsey Theory on Countable Sets

II. Devlin's Theorem for colorings of $[\mathbb{Q}]^m$.

III. Classic Methodology for characterizing the big Ramsey degrees of $(\mathbb{Q}, <)$.

   (a) Milliken's Ramsey Theorem for Strong Trees
   (b) Diagonal Antichains and Strong Tree Envelopes
   (c) Upper Bounds
   (d) Lower Bounds

IV. The Halpern-Läuchli Theorem

   (a) Harrington's 'forcing proof'
   (b) Halpern-Läuchli as Pigeonhole for inductive proof of Milliken

Day 2: Forcing on Coding trees and general big Ramsey degree theory

Day 3: Infinite-dimensional Ramsey theory

I. Ramsey Theory on Countable Sets

Partition Theorems on finite subsets of ω

Theorem (Pigeonhole Principle (PP))

If infinitely many marbles are partitioned into finitely many buckets, then some bucket contains infinitely many marbles.

Theorem (Ramsey)

Given m, r *and a coloring* $\chi : [\mathbb{N}]^m \to r$, *there is an infinite subset* $N \subseteq \mathbb{N}$ *such that* χ *takes one color on* $[N]^m$.

PP $=$ RT with $m = 1$.

Inductive Proof of Ramsey's Theorem using PP

<u>Base Case</u>: $m=1$. Pigeonhole Principle.

<u>Ind Hyp</u>: Ramsey's Theorem holds on $[\omega]^m$.

<u>Ind Step</u>: Let $c:[\omega]^{m+1}\to r$ be given.

Let $\prec$ be the well-ordering on $[\omega]^m$ defined as follows: For $s=\{i_0<i_1<\dots<i_{m-1}\}$, $t=\{j_0<j_1<\dots<j_{m-1}\}$,

$$s\prec t \iff \begin{cases}\text{either } i_{m-1}<j_{m-1}\\ \text{or } i_{m-1}=j_{m-1} \text{ and } \{i_0,\dots,i_{m-2}\}<_{lex}\{j_0,\dots,j_{m-2}\}.\end{cases}$$

Note: $\prec$ well orders $[\omega]^{m+1}$ in order type ω.

Inductive Proof of Ramsey's Theorem using PP

Let $\langle s_n : n<\omega\rangle$ enumerate $[\omega]^m$ in $\prec$-increasing order.
The Ind. Step is now proved via induction on the seq $\langle s_n\rangle$.

By PP, $\exists M_0 \in [\omega \setminus \max(s_0)+1]^\omega$ and a color $r_0 \in r$ s.t. $c(s_0 \cup \{j\}) = r_0$, $\forall j \in M_0$

By PP, $\exists M_1 \in [M_0 \setminus \max(s_1)+1]^\omega$ and a color $r_1 \in r$ s.t. $c(s_1 \cup \{j\}) = r_1$, $\forall j \in M_1$ ...

Now proceed with general n step of the induction.

Let $m_n = \min(M_n)$ and $N = \{m_n : n<\omega\}$.

Apply Ind Hyp to $[N]^m$ to get $P \in [N]^\omega$ with all r_i = same ℓ, $\forall s_i \in [P]^m$. Argue that $[P]^{m+1}$ is monochromatic for c with color ℓ.

Inductive Proof of Ramsey's Theorem using PP

Recap: Proof Structure:

Ind on m: Base Case $m=1$. Pigeonhole.

Ind Hyp: Assume Theorem true for m.

Ind Step: Order $[\omega]^m$ in order type ω so that it is a sequence of the sort

{Finitely many things with max 0} $<$ {Finitely many things with max 1} $<$ {Finitely many things with max 2} $<$ . . .

In each block do a finite induction using PP.
Between the blocks is an infinite induction.

Final Step: Apply Ind Hyp.

Which infinite structures carry analogues of Ramsey's Theorem?

We will discuss this tomorrow.

Today, we thoroughly investigate the rationals as a dense linear order.

II. Devlin's Theorem for colorings of $[\mathbb{Q}]^m$.

The Rationals as a Dense Linear Order

- $(\mathbb{Q}, <)$ has a Pigeonhole Principle. (indivisible)
- Ramsey's Theorem fails for pairs of rationals. (Sierpiński, 1933)

Key Idea: Enumerate $\mathbb{Q}$ as $\langle q_0, q_1, q_2, \ldots \rangle$

Define a coloring : for $i < j$, $c(\{q_i, q_j\}) = \begin{cases} \text{red} & \text{if } q_i < q_j \\ \text{blue} & \text{if } q_j < q_i \end{cases}$

$-3/2$	-1	$-1/2$	0	$1/2$	$3/4$	1	$3/2$
q_2	q_7	q_1	q_0	q_3	q_6	q_5	q_4

These patterns are unavoidable.

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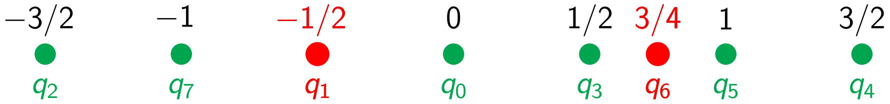

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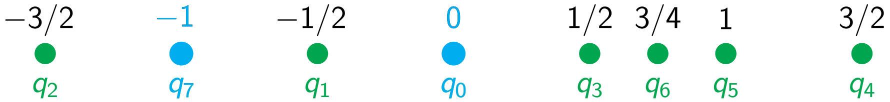

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Coloring Finite Sets of Rationals

Theorem (D. Devlin, 1979)

Given m, if $[\mathbb{Q}]^m$ is colored by finitely many colors, then there is a subcopy $\mathbb{Q}' \subseteq \mathbb{Q}$ forming a dense linear order such that $[\mathbb{Q}']^m$ take no more than $C_{2m-1}(2m-1)!$ colors. This bound is optimal.

m	**Bound**
1	1
2	2
3	16
4	272

C_i is from

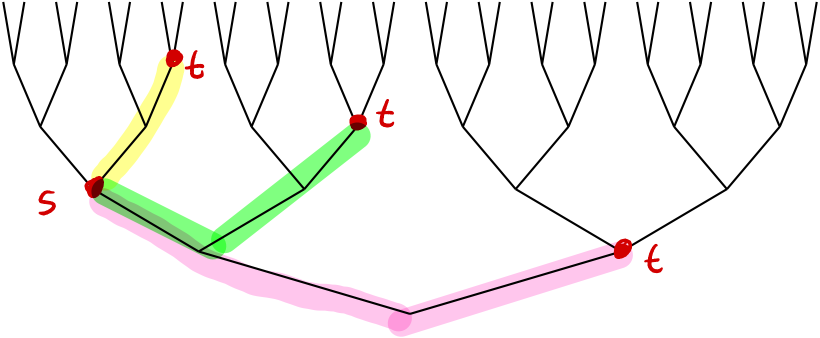
$$\tan(x) = \sum_{i=0}^{\infty} C_i x^i$$

- Galvin (1968) The bound for pairs is two.
- Laver (1969) Upper bounds for all finite sets.

III. Classic Methodology for characterizing the big Ramsey degrees of $(\mathbb{Q}, <)$.

(a) Representing $\mathbb{Q}$ by $2^{<\omega}$
(b) Milliken's Ramsey Theorem for Strong Trees
(c) Diagonal Antichains
(d) Strong Tree Envelopes
(e) Upper Bounds
(f) Lower Bounds

III(a). $2^{<\omega}$ represents $(\mathbb{Q}, <)$



$$s < t \iff s \supseteq (s \wedge t)^\frown 0 \text{ or } t \supseteq (s \wedge t)^\frown 1$$

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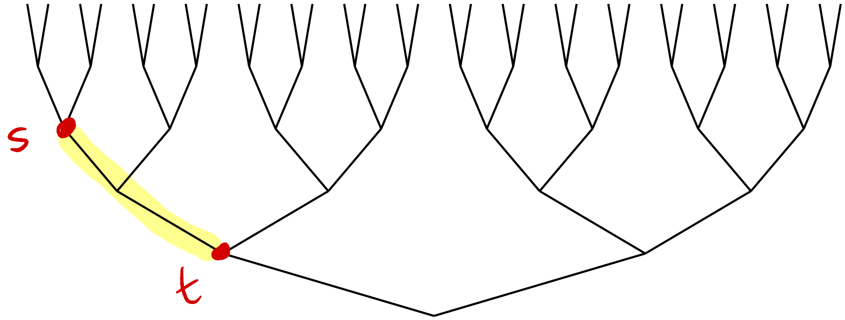
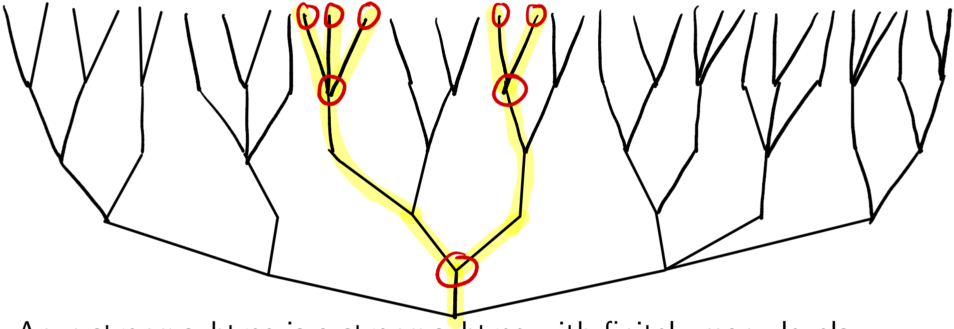


$$s < t \iff s \supseteq (s \wedge t)^\frown 0 \text{ or } t \supseteq (s \wedge t)^\frown 1$$

There are 4 configurations in $2^{<\omega}$ for pairs $s<t$.

Let T be a finitely branching subtree of $\omega^{<\omega}$ with no terminal nodes. $S \subseteq T$ is a **strong subtree** of T if there is a set $A \subseteq \omega$ of levels such that each node in T of length $k \in A \setminus \max(A)$ branches maximally in T and each node in T of length $k \notin A$ does not branch.



An n-strong subtree is a strong subtree with finitely many levels.

A 3-strong subtree.

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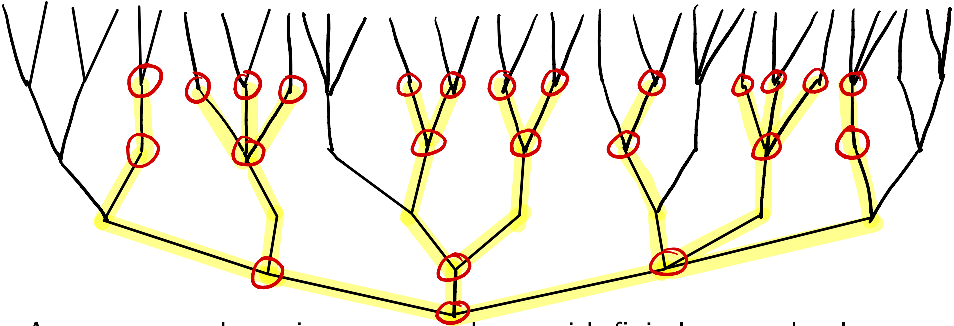

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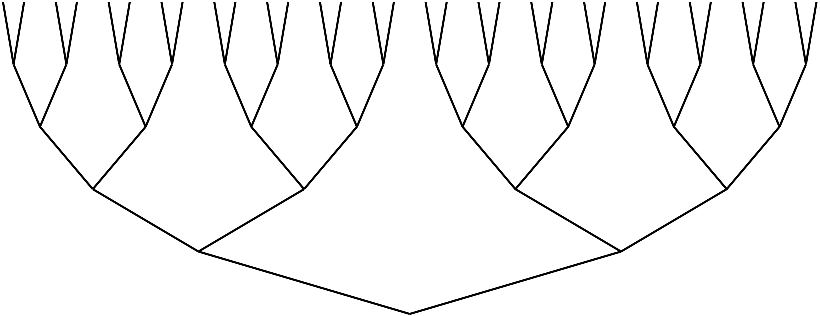
A 4-strong subtree.

Milliken's Theorem

Theorem (Milliken, 1979)

Let T be a finitely branching subtree of $\omega^{<\omega}$ with no terminal nodes. Given $n \geq 1$ and a coloring of all n-strong subtrees of T into finitely many colors, there is an infinite strong subtree of T in which all n-strong subtrees have the same color.



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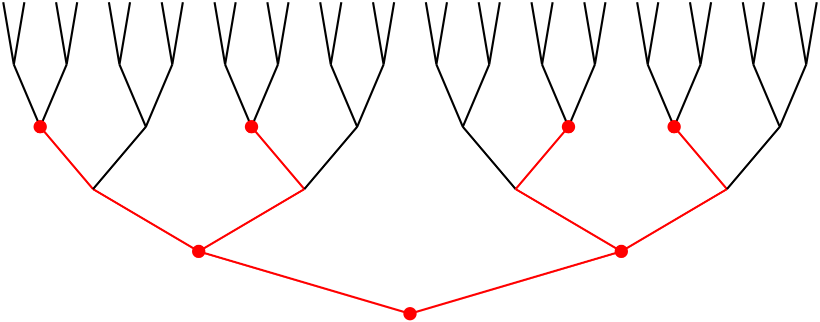

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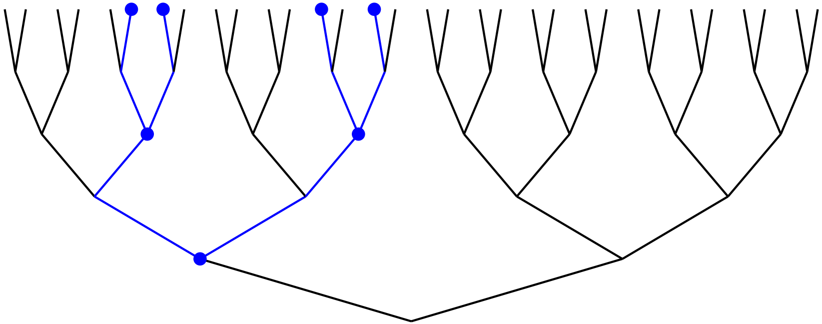


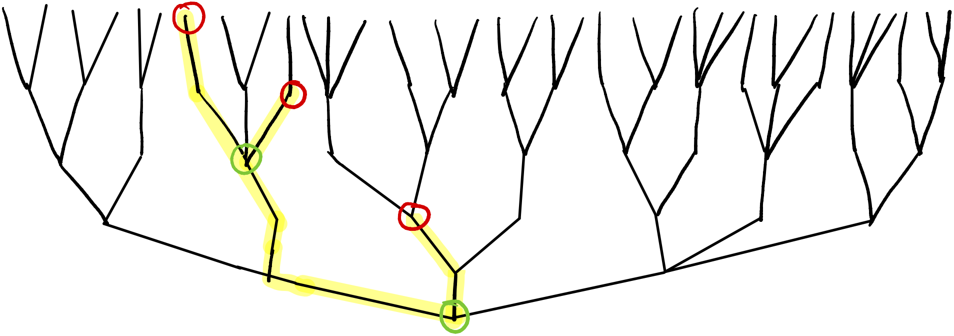
III(c). Diagonal Antichains

A subset $A \subseteq T$ is an **antichain** if each pair of nodes in A is incomparable in the tree ordering.

An antichain $A \subseteq T$ is **diagonal** if its meet closure $\text{cl}(A)$ has the following properties:

1. Any two distinct meets occur on different levels.
2. Each meet has exactly two immediate successors.

III(c). Diagonal Antichains



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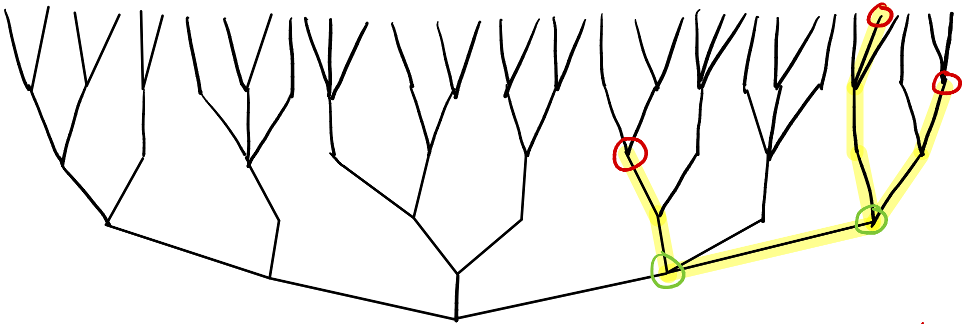
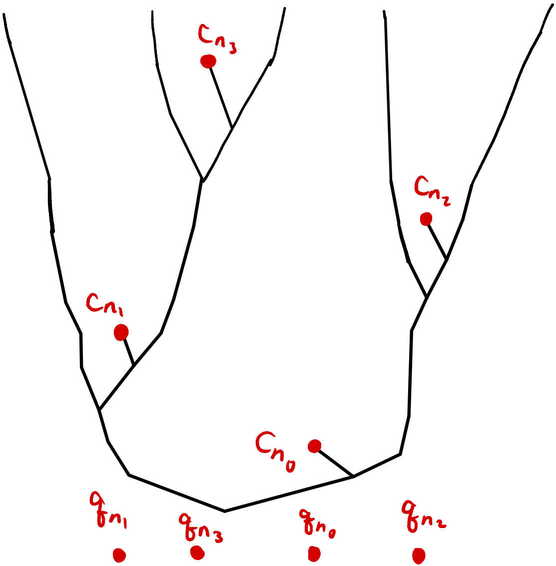


Note: Unavoidable colorings will take into account # of terminal nodes, relative lengths of terminal nodes and meet nodes, lex order.

III(c). There is a diagonal antichain representing $\mathbb{Q}$



III(d). Strong Tree Envelopes

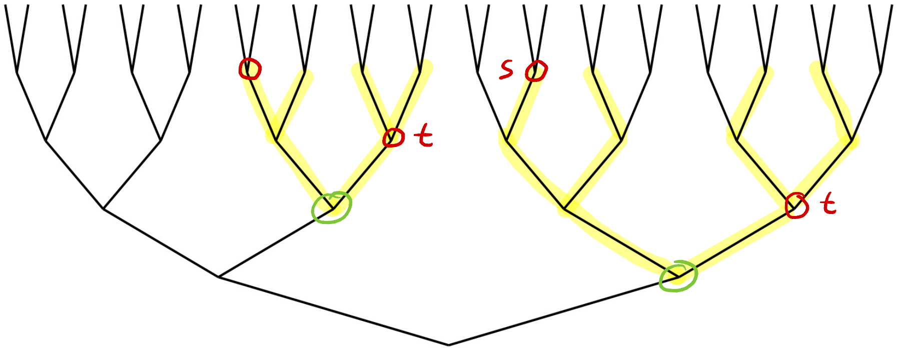
Let $T \subseteq \omega^{<\omega}$ be a finitely branching tree with no terminal nodes.

Let $A \subseteq T$ be a finite antichain.

Let n be the number of levels in the meet closure $\mathrm{cl}(A)$ of A.

A **strong tree envelope** of A is an n-strong subtree of T containing A.

III(d). Strong Tree Envelopes in $2^{<\omega}$



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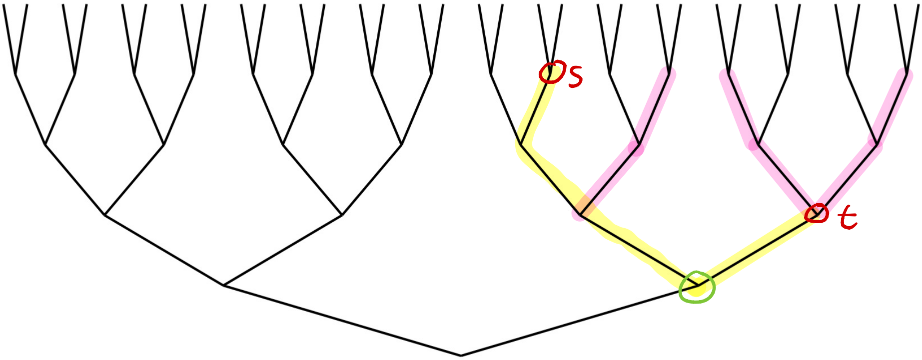
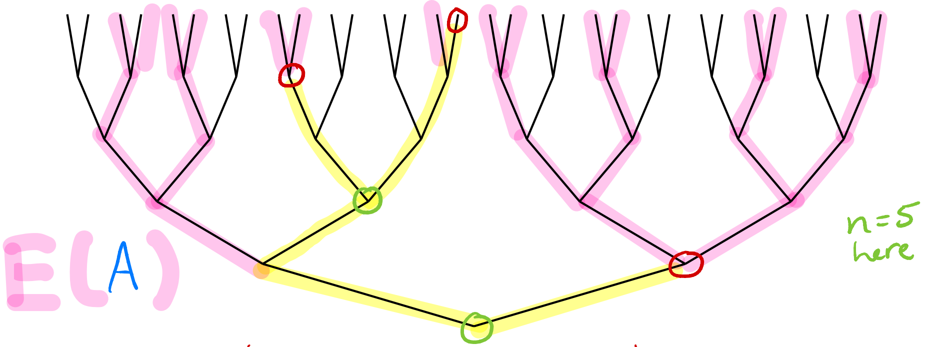


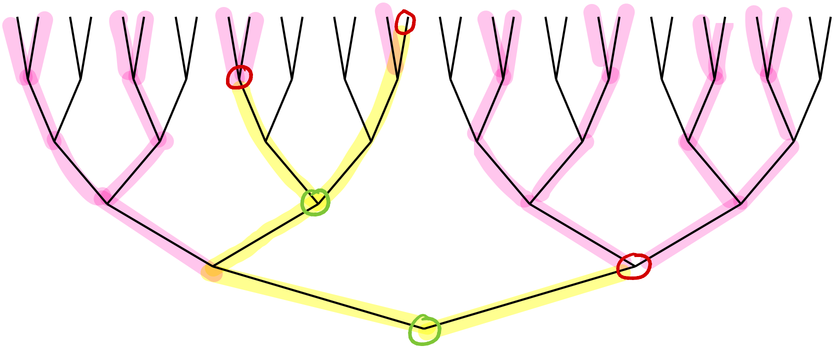
Fix a finite diagonal antichain $A \subseteq 2^{<\omega}$.
Let $n = \#$ of levels in the meet closure of A.



An envelope of A, $E(A)$, is an n-strong subtree of $2^{<\omega}$ which contains A.

III(e). Upper Bound Proof using Milliken and envelopes

Fix a finite diagonal antichain $A \subseteq 2^{<\omega}$.

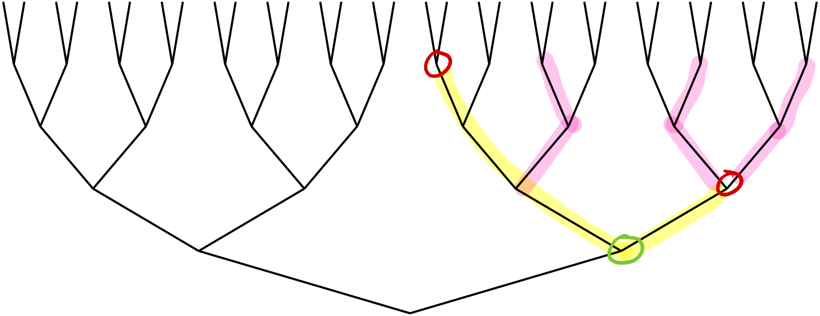


Note: There can be more than one envelope of A, but there are only finitely many.

III(e). Upper Bound Proof using Milliken and envelopes

$\mathcal{U} \subseteq \omega^{<\omega}$ finitely branching tree with no leaves.

$\mathcal{S}_\infty(\mathcal{U})$ = space of all infinite strong subtrees of $\mathcal{U}$.



$\mathcal{S}_n(\mathcal{U})$ = set of all n-strong subtrees of T.

III(e). Upper Bound Proof using Milliken and envelopes

Given a finite diagonal antichain $A \subseteq 2^{<\omega}$,
Let n = number of levels in meet closure of A.

Lemma: Given any n-strong subtree $S \subseteq 2^{<\omega}$ there is exactly one isomorphic subcopy of A in S. Pf: Exercize. Or see Lemma 6.12 [Todorcevic].

Let c color all copies of A in $2^{<\omega}$ into r colors.
Transfer this coloring to $\mathcal{S}_n(2^{<\omega})$:

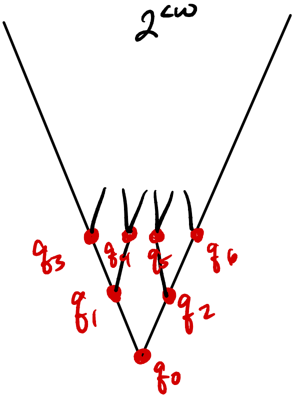
Given $S \in \mathcal{S}_n(2^{<\omega})$ let $d(S)$ be the color of the copy of A in S. Apply Milliken's Theorem to get $T \in \mathcal{S}_\infty(2^{<\omega})$ s.t. $\mathcal{S}_n(T)$ is monochr for d.

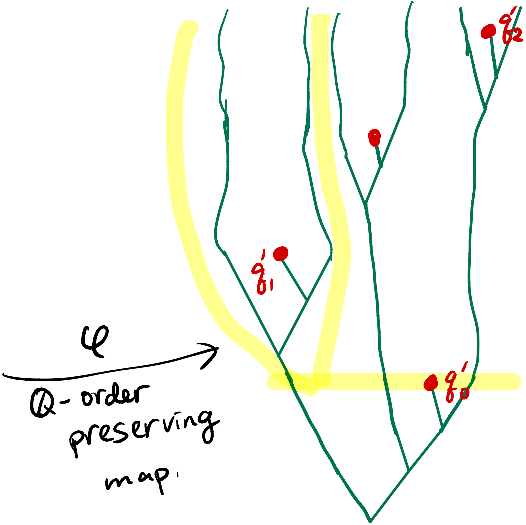
Then all copies of A in T have same color.

Now, given m and $c : [\mathbb{Q}]^m \longrightarrow r$, enumerate the diagonal antichains of size m as $A_0, \ldots, A_k$. Use Milliken's Theorem to get $2^{<\omega} \supseteq T_0 \supseteq \ldots \supseteq T_k$ all in $\mathcal{S}_\infty(2^{<\omega})$ so that all copies of A_i in T_i have the same color. In fact, $\forall i \le k$, all copies of A_i in T_k have the same color.

III(e). Upper Bound Proof using Milliken and envelopes

Since there is a diagonal antichain $\Delta \subseteq 2^{<\omega}$ representing $\mathbb{Q}$, k is an upper bound for the big Ramsey degree of m-sized linear orders in $\mathbb{Q}$.





Look for "large" sets.

III(f). Lower Bound Proof

References for material so far:

D. Devlin, PhD Thesis, 1979 Dartmouth

Todorcevic, "Intro. to Ramsey Spaces" Chapters 3 and 6

Laflamme-Sauer-Vuksanovic 2006 has an accessible lower bound proof for Rado graph which can be simplified for $\mathbb{Q}$.

IV. The Halpern-Läuchli Theorem

Halpern-Läuchli Theorem - strong tree version

Notation: $$\bigotimes_{i<d} T_i := \bigcup_{n<\omega} \prod_{i<d} T_i(n)$$

Theorem (Halpern-Läuchli, 1966)

Let $T_i \subseteq \omega^{<\omega}$, $i < d$, be finitely branching trees with no terminal nodes. Given a coloring $\chi : \bigotimes_{i<d} T_i \to 2$, there are strong subtrees $S_i \leq T_i$ with nodes of the same lengths such that χ is constant on $\bigotimes_{i<d} S_i$.

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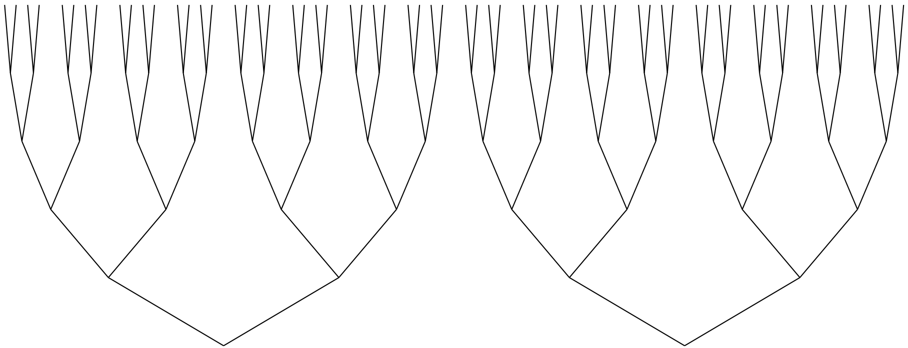
HL was distilled as a key lemma in the proof that the Boolean Prime Ideal Theorem is strictly weaker than the Axiom of Choice over ZF. (Halpern-Lévy, 1971)

Example: Coloring $T_0 \otimes T_1$



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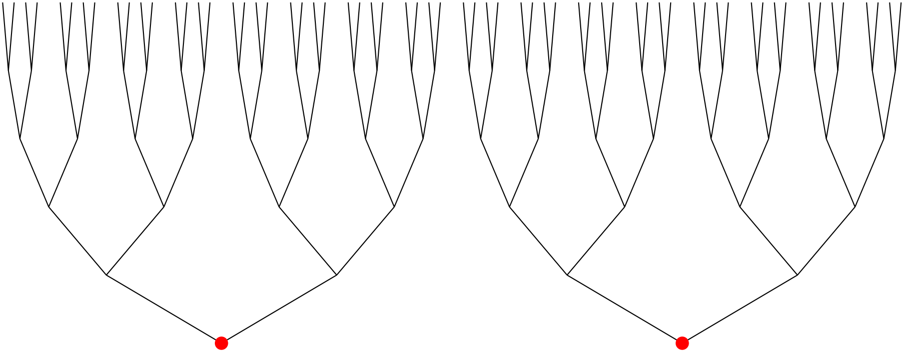

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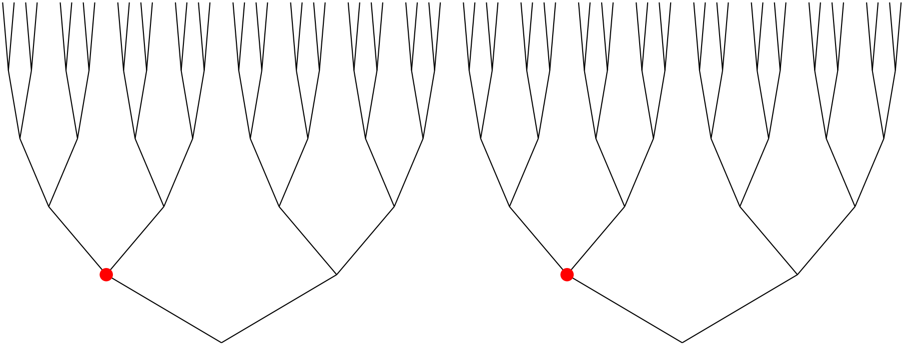

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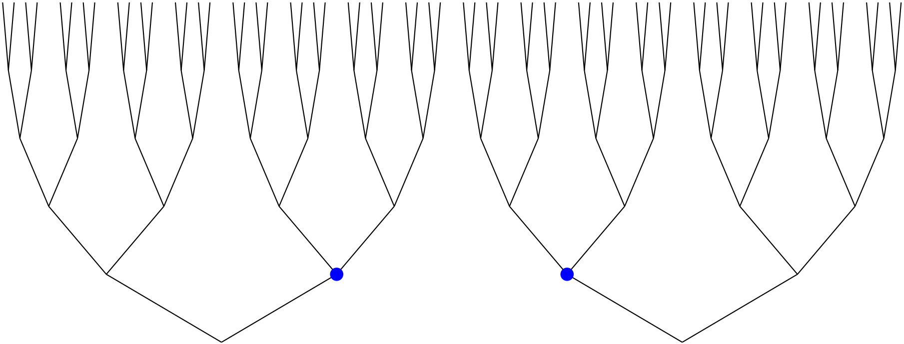

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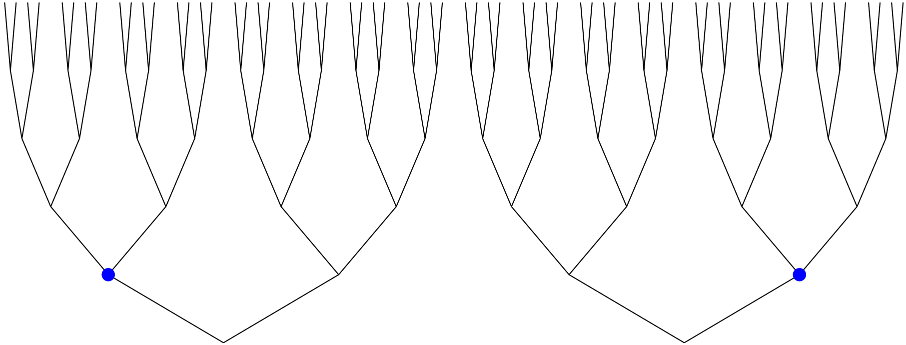

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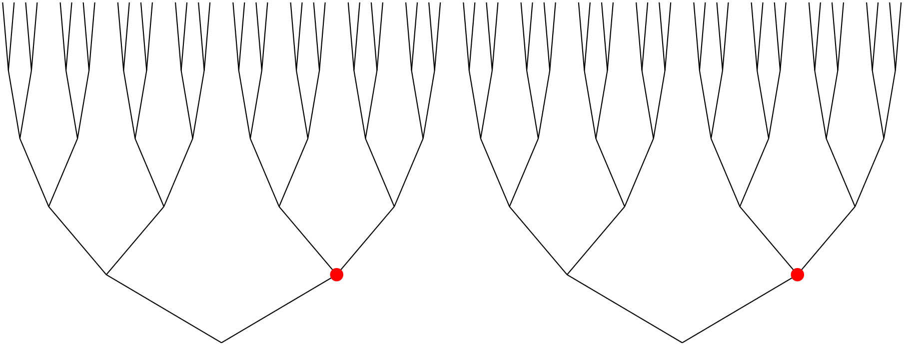

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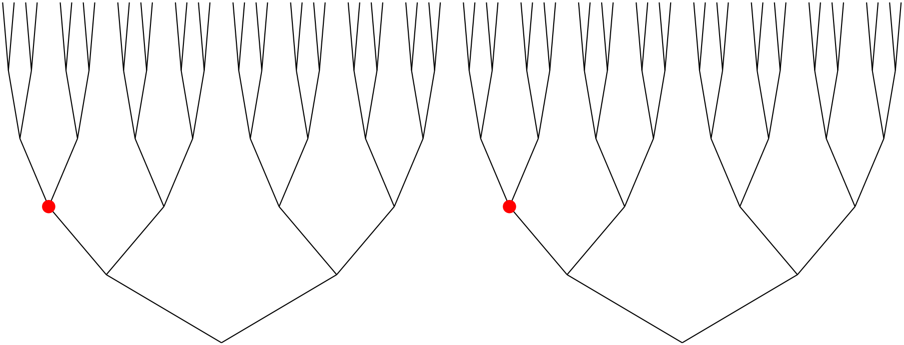

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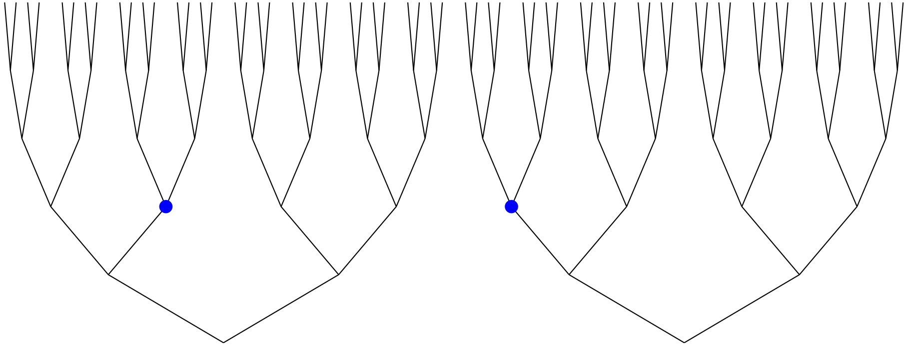

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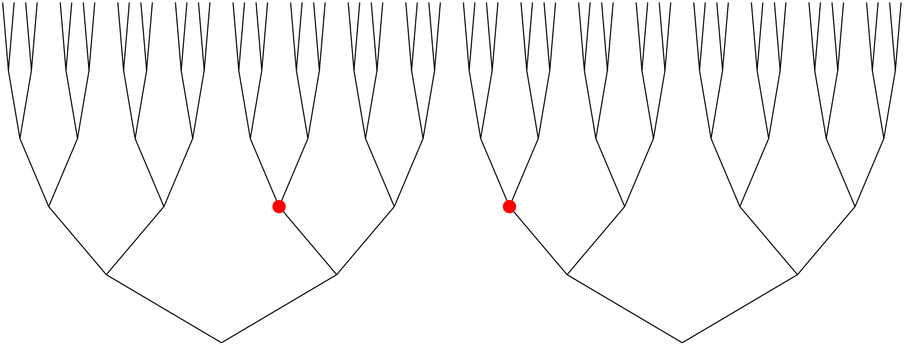

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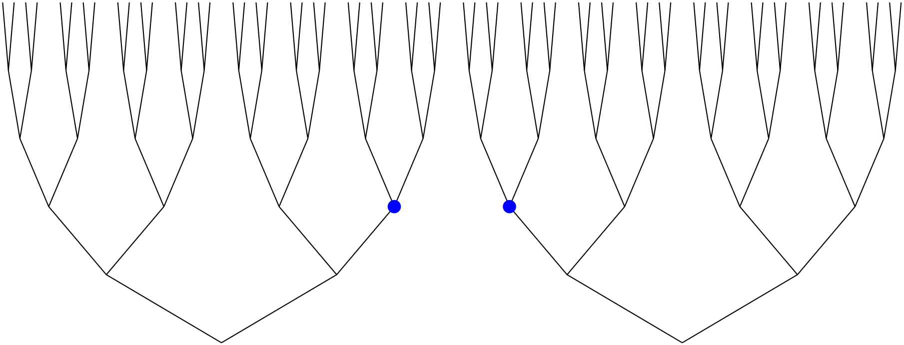

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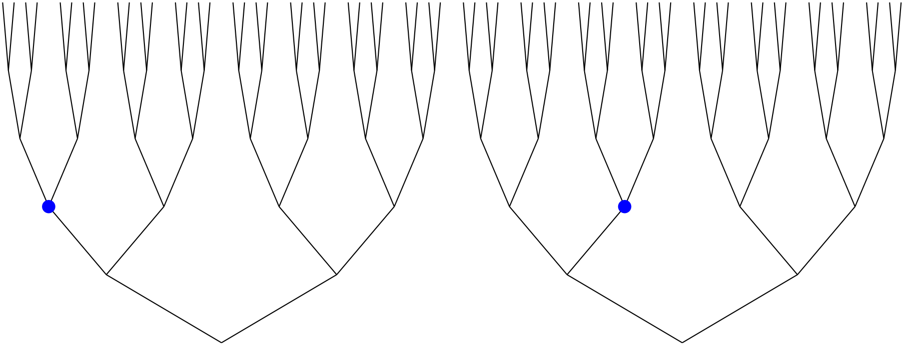

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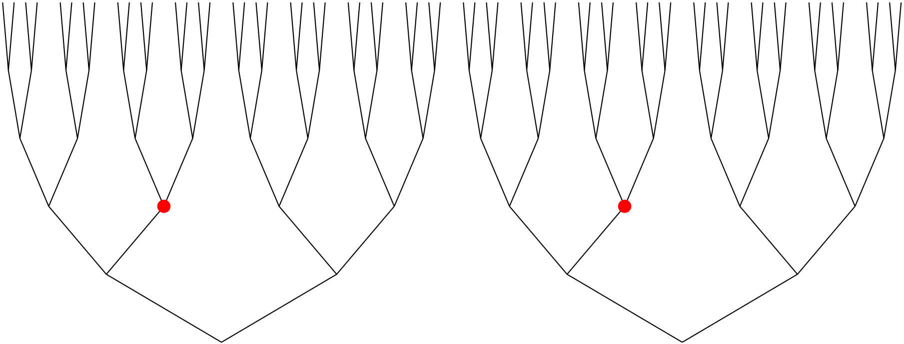
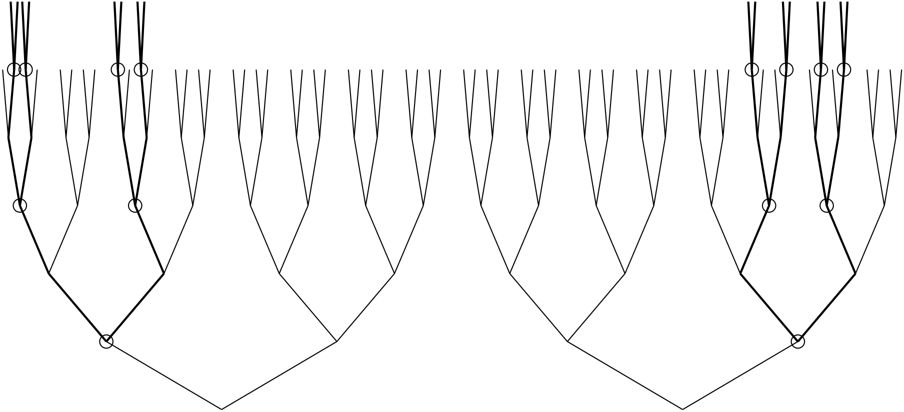


HL gives $S_i \subseteq T_i$ with one color on $S_0 \otimes S_1$



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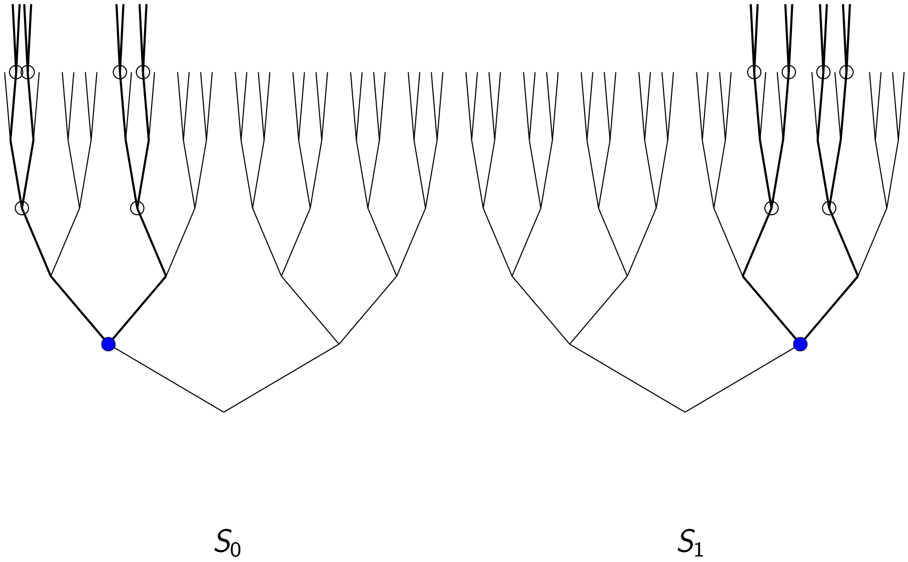

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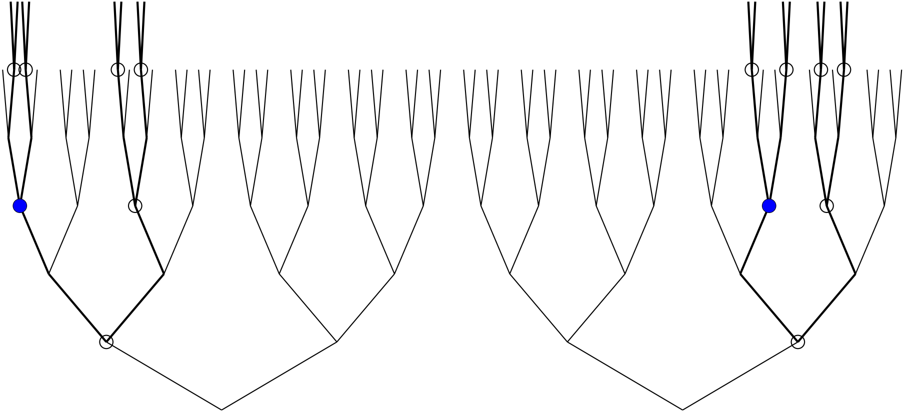

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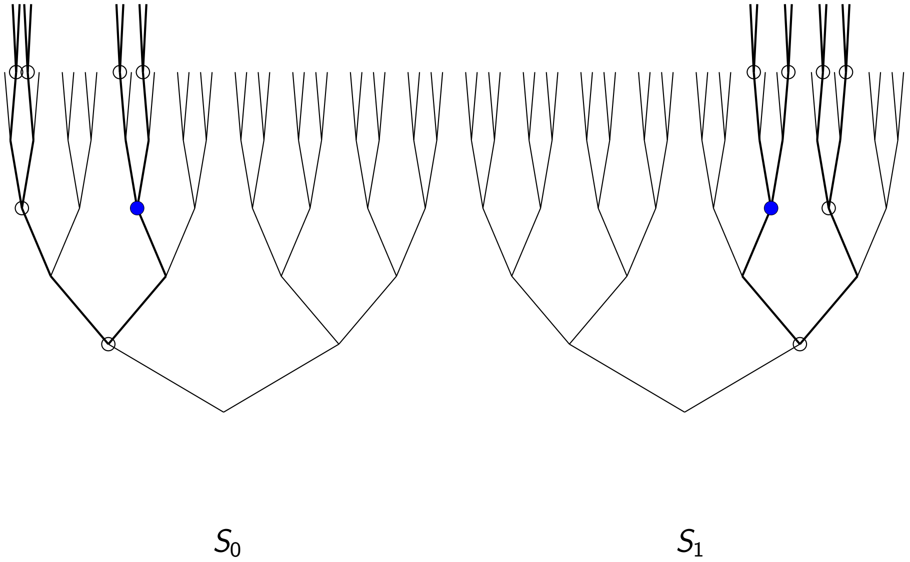

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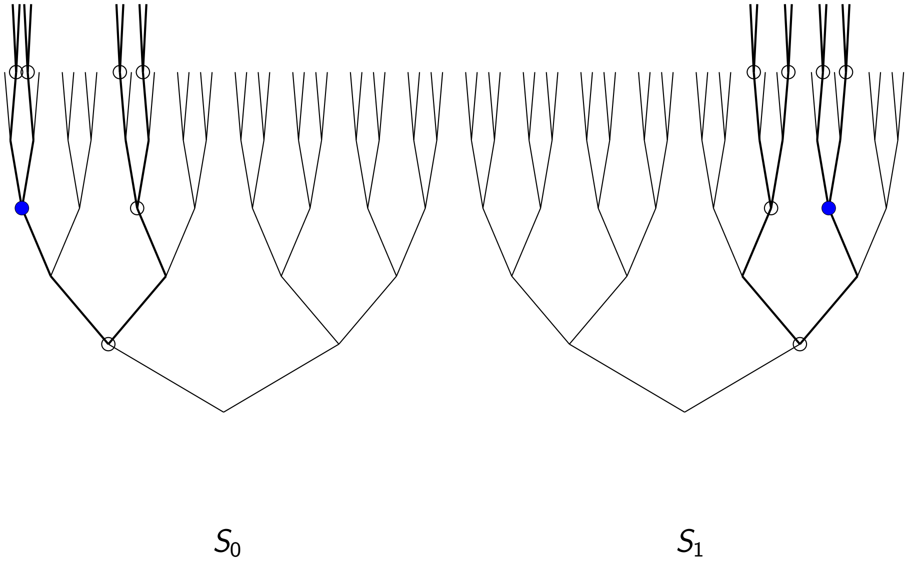

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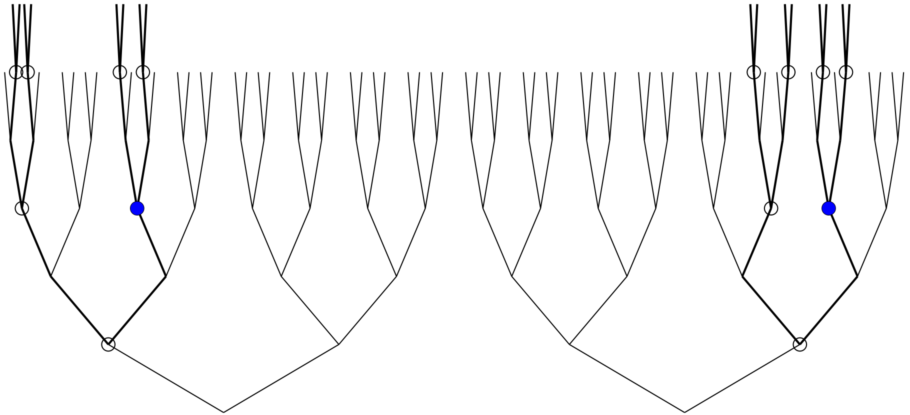
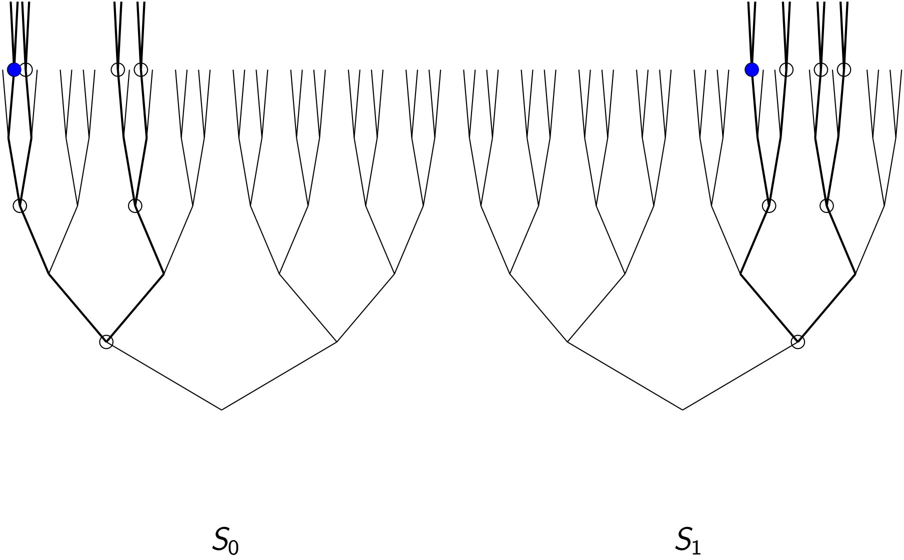
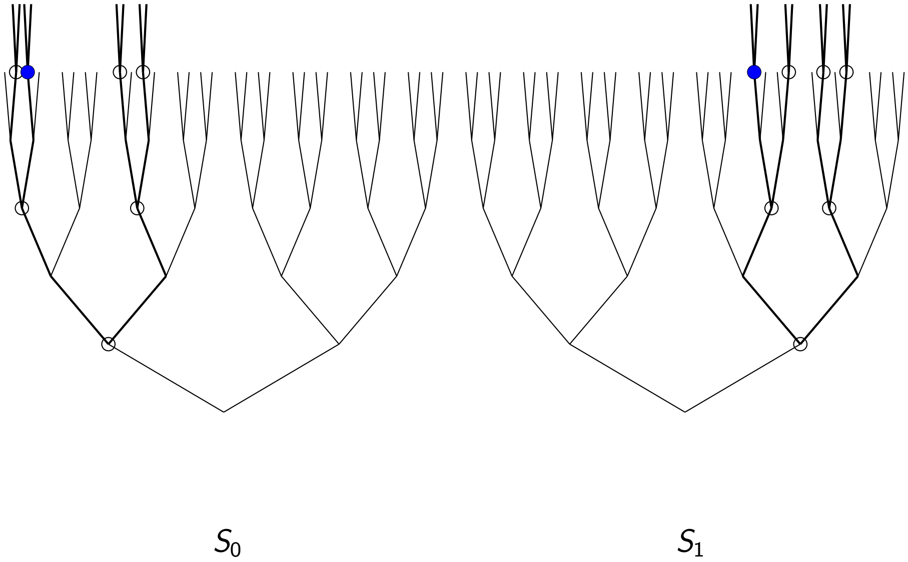


S_0 S_1

HL gives $S_i \subseteq T_i$ with one color on $S_0 \otimes S_1$



HL gives $S_i \subseteq T_i$ with one color on $S_0 \otimes S_1$



IV(a). Harrington's 'Forcing' Proof of Halpern-Läuchli Theorem

Harrington devised a proof of the Halpern–Läuchli Theorem that uses forcing methods to do countably many searches for finite objects.

This is NOT an absoluteness proof; no generic extensions involved.

References:

Farah and Todorcevic, *Some applications of the method of forcing*, Yenisei Series, 1995.

Dobrinen, *Forcing in Ramsey theory*, RIMS Kokyuroku (2017) and

Dobrinen, *The Ramsey theory of Henson graphs*, JML 2023, Section 3.4 (with fewer typos than 2017).

Thanks to Laver for an outline of this proof in 2011!

Harrington's 'Forcing' Proof of HL

Fix $d \geq 2$ and let $T_i = 2^{<\omega}$ $(i < d)$ be finitely branching trees with no terminal nodes. Fix a coloring $c : \bigotimes_{i<d} T_i \to 2$.

Let $\kappa = \beth_{2d}$. Then $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$. (Erdős–Rado)

$\mathbb{P} =$ Cohen forcing adding κ new branches to each tree T_i, $i < d$.

$\mathbb{P}$ is the set of functions p of the form

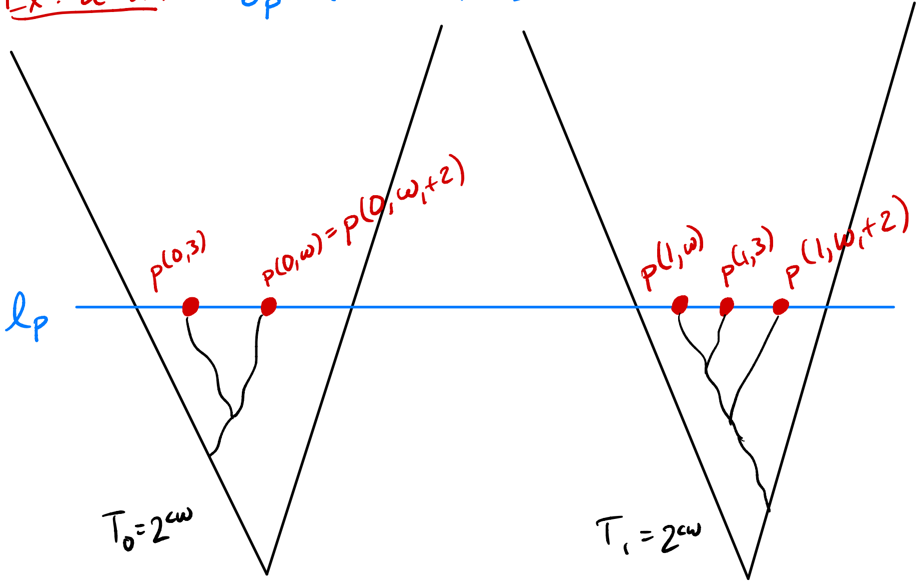
$$p : d \times \vec{\delta}_p \to \bigcup_{i<d} T_i \restriction \ell_p$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$, $\ell_p < \omega$, and $\forall i < d$, $\{p(i,\delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \restriction \ell_p$.

$q \leq p$ iff $\ell_q \geq \ell_p$, $\vec{\delta}_q \supseteq \vec{\delta}_p$, and $\forall (i,\delta) \in d \times \vec{\delta}_p$, $q(i,\delta) \supseteq p(i,\delta)$.

Harrington's 'Forcing' Proof of HL

Ex : $d=2$. $\vec{\delta}_p = \{3, \omega, \omega_1+2\}$



Harrington's 'Forcing' Proof: Set-up for the Ctbl Coloring

For $i < d$, $\alpha < \kappa$, $\dot{b}_{i,\alpha}$ denotes the α-th generic branch in T_i.

$$\dot{b}_{i,\alpha} = \{\langle p(i,\alpha), p\rangle : p \in \mathbb{P}, \text{ and } (i,\alpha) \in \mathrm{dom}(p)\}.$$

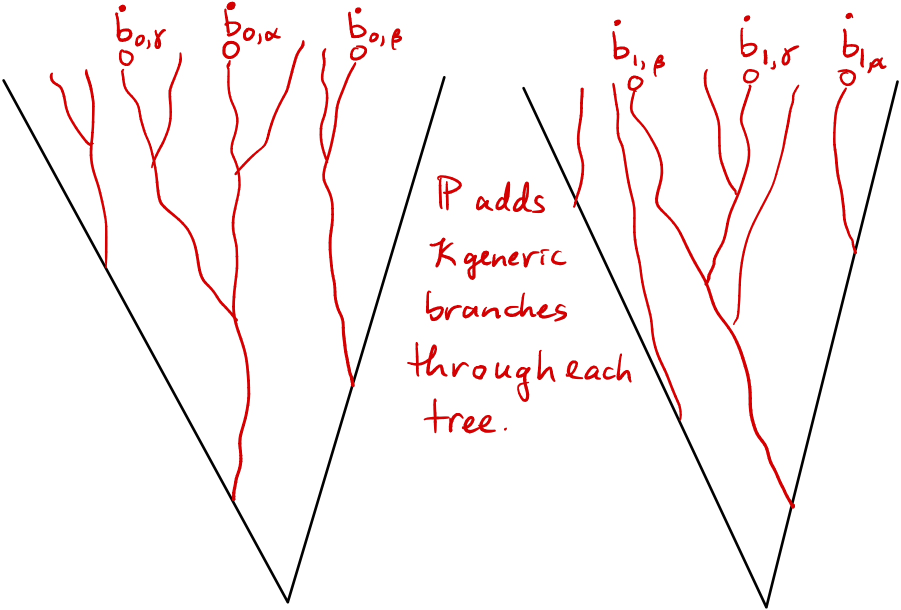
Note: If $(i,\alpha) \in \mathrm{dom}(p)$, then $p \Vdash \dot{b}_{i,\alpha} \restriction \ell_p = p(i,\alpha)$.

Let $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name for a non-principal ultrafilter on ω.

For $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_{d-1}\rangle \in [\kappa]^d$, let $\dot{b}_{\vec{\alpha}} := \langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d-1,\alpha_{d-1}}\rangle$.

Let $\dot{b}_{\vec{\alpha}} \restriction \ell := \{\dot{b}_{i,\alpha_i} \restriction \ell : i < d\}$.

Harrington's 'Forcing' Proof



Harrington's 'Forcing' Proof

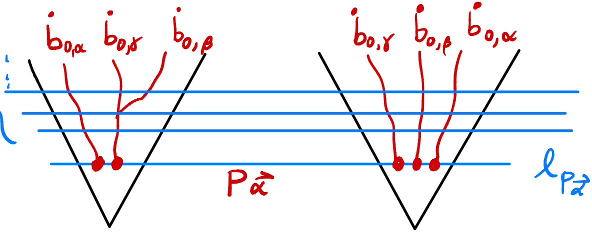
GOAL: Find infinite sets $K_0 < K_1 < \dots < K_{d-1}$, subsets of κ, and a set of conditions $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ which are compatible, have the same images in T, and so that for some $\varepsilon^* < 2$, there are $\dot{\mathcal{U}}$-many ℓ for which $h(\dot{b}_{\vec{\alpha}} \restriction \ell) = \varepsilon^*$.

Then we will let $t_i^* = p_{\vec{\alpha}}(i, \alpha_i)$ for any/all $\vec{\alpha} \in \prod_{i<d} K_i$.

These t_i^*, $i < d$, will be the starting nodes above which we will build the subtrees satisfying HL.

Harrington's 'Forcing' Proof

Set of levels ℓ where $c(\dot{b}_{\vec\alpha}\restriction\ell)=\varepsilon_{\vec\alpha}$ is in $\dot{\mathcal{U}}$.



Towards the GOAL:

For $\vec{\alpha} \in [\kappa]^d$, take some $p_{\vec\alpha} \in \mathbb{P}$ with $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec\alpha}}$ such that

1. $p_{\vec\alpha}$ decides an $\varepsilon_{\vec\alpha} \in 2$ s.t. $p_{\vec\alpha} \Vdash c(\dot{b}_{\vec\alpha} \restriction \ell) = \varepsilon_{\vec\alpha}$ for $\dot{\mathcal{U}}$ many ℓ,
2. $c(\{p_{\vec\alpha}(i, \alpha_i) : i < d\}) = \varepsilon_{\vec\alpha}$.

For $\vec{\theta} \in [\kappa]^{2d}$ and $\iota : 2d \to 2d$, let

$$\vec{\alpha} = (\theta_{\iota(0)}, \theta_{\iota(2)}, \ldots, \theta_{\iota(2d-2))}) \text{ and } \vec{\beta} = (\theta_{\iota(1)}, \theta_{\iota(3)}, \ldots, \theta_{\iota(2d-1)}).$$

Define
$$f(\iota, \vec{\theta}) = \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, \langle\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle : i < d\rangle,$$
$$\langle\langle i, j\rangle : i < d,\ j < k_{\vec{\alpha}}, \delta_{\vec{\alpha}}(j) = \alpha_i\rangle,$$
$$\langle\langle j, k\rangle : j < k_{\vec{\alpha}},\ k < k_{\vec{\beta}},\ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)\rangle\rangle,$$

where $k_{\vec{\alpha}} = |\vec{\delta}_{p_{\vec{\alpha}}}|$, and $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}}\rangle$ enumerates $\vec{\delta}_{p_{\vec{\alpha}}}$.

Define $f(\vec{\theta}) = \langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I}\rangle$.

Harrington's 'Forcing' Proof: Set of compatible conditions

$\kappa \rightarrow (\aleph_1)^{2d}_{\aleph_0}$ implies $\exists H \in [\kappa]^{\aleph_1}$ homogeneous for f.

Take $K_i \in [H]^{\aleph_0}$ where $K_0 < \cdots < K_{d-1}$ and let $K := \bigcup_{i<d} K_i$.

Main Lemma. $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

Harrington's 'Forcing' Proof

The Main Lemma proceeds via some smaller lemmas.

A key idea used in a lot of Ramsey Theory is the sliding property of indiscernibles.

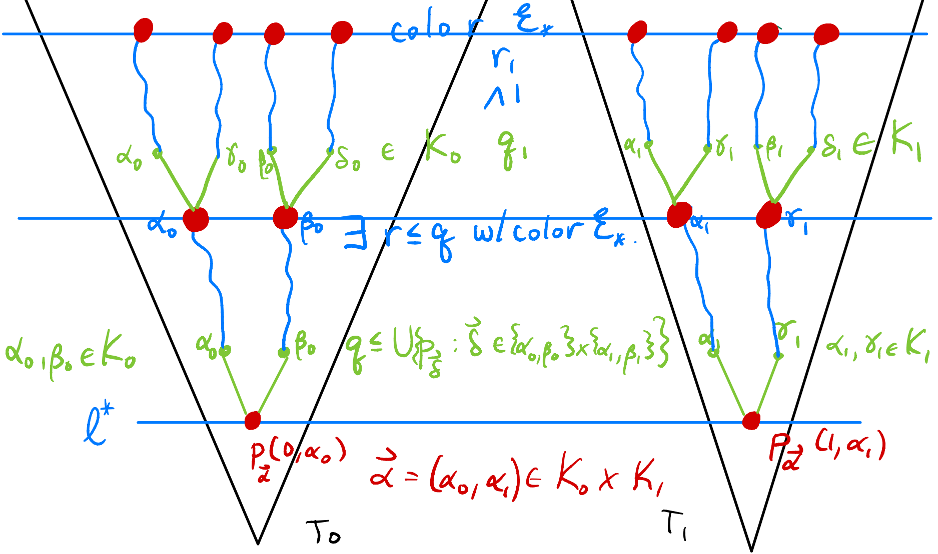
If i k equivalent

and j k equivalent,

then i j equivalent.

Building the monochromatic subtrees

Proof uses ω applications of the forcing mechanism to get a level set extension with color ε_*.



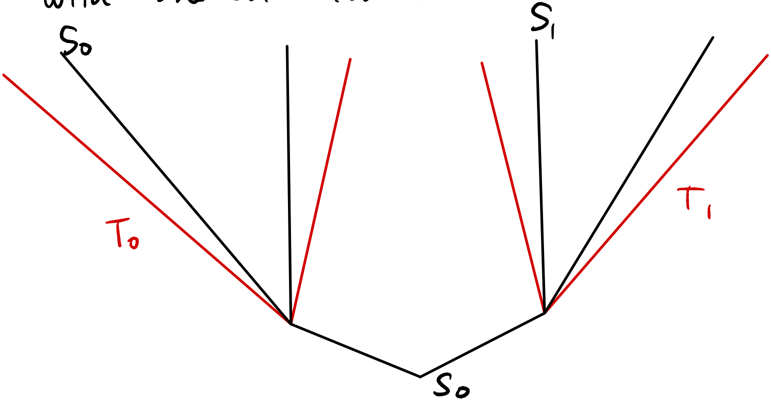
Pf by induction on n.

Base: $n=1$. 1-strong subtrees are singleton nodes.

HL is Milliken for 1-strong trees

n = 2: Fix $s_0 = \langle\rangle$. s_0 has immediate successors $\langle 0 \rangle$ and $\langle 1 \rangle$.

Apply HL for 2 trees to get trees S_0, S_1 with one color for level products.



IV(b). HL as Pigeonhole for inductive proof of Milliken

Continue up the tree in finite blocks (next level of the current subtree). Like RT.

At end of this infinite induction, we transfer the coloring to singleton nodes.

Last step, apply Ind Hyp (Milliken for 1-strong) and get a strong subtree in which all 2-strong subtrees have same color.

IV(b). HL as Pigeonhole for inductive proof of Milliken

Excercize - you write out general inductive proof for (n+1)-strong trees assuming Milliken for n-strong trees.

Remarks

- Harrington's forcing proof of Halpern-Läuchli along with the development of coding trees opened the door to proving the Henson graphs have finite big Ramsey degrees, which in turn, inspired a rapid expansion of results and methods.

- In their AMS Memoirs book (2023), Anglès d'Auriac, Cholak, Dzhafarov, Monin, and Patey, the Halpern-Läuchli Theorem is computably true and admits strong cone avoidance.